Thickness driven stabilization of saw-tooth–like domains upon phase transitions in ferroelectric thin films with depletion charges

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Ionized impurities have nearly always been neglected in discussing the limit of functionality of ferroelectric thin films. One would certainly expect that the thickness limit for functionality would be altered in the presence of ionized impurities, but how this would occur remains unclear. In this article, we analyze the domain structures as well as the phase transition temperatures in films with depletion charges for various film thicknesses. Depletion charges induce a position-dependent built-in field that leads to an inhomogeneous distribution of ferroelectric polarization. Such an inhomogeneity in the polarization results in strong depolarizing fields in films. We show that formation of saw-tooth–type domains is a way to circumvent the depolarizing fields, even in films with ideal electrodes. There is a critical film thickness above which the saw-tooth domains develop. On the other hand, the phase transition of the ultrathin structures with electrodes having a finite screening length, namely real electrodes, is always into the multidomain state during cooling from the paraelectric state, regardless of the presence of depletion charges. An important finding we have is that the transition temperature in films with real electrodes does not depend nearly at all on the depletion charge density unless it is very high (\(>10^{26}\) ionized impurities/m\(^3\)). Relatively thick films (>8 nm in this work) with real electrodes that have very high depletion charge densities have transition temperatures very similar to those with the same charge density, but with ideal electrodes, making us conclude that thick films with high depletion charge densities will hardly feel the finite screening effects. The results are provided for (001) BaTiO\(_3\) films grown on (001) SrTiO\(_3\) substrates with pseudomorphic top and bottom metallic electrodes. © 2012 American Institute of Physics. [http://dx.doi.org/10.1063/1.3691938]
crucial to elaborate the interaction between depletion charges and the consequences of the extent of screening at the film-electrode interface. Film thicknesses at the order of a few tens of nanometers are comparable to the depletion zone widths expected in ferroelectric films, with metallic electrodes inducing a depletion potential of around 1 V. For typical densities of impurities (between $10^{24}$ and $2 \times 10^{26}$) we consider in this paper, we estimate depletion widths of around 100 nm to around 8 nm at each electrode interface. As the total of these depletion width values at each interface are quite close to the film thickness ranges of interest in our work, we assume fully depleted films when stated, particularly when finding the phase transition temperatures. The discussion on the effect of partial depletion on the domain structures is an exception to the full depletion assumption, but we show that the two situations are not very different for films not thicker than 28–30 nm. Depletion widths of around 30 nm and ionized impurity densities of around $10^{25}$–$10^{27}$/m$^3$ have been reported by Pintilie et al. using interfacial capacitance measurements for PbZr$_{0.5}$Ti$_{0.5}$O$_3$ films.$^{36,37}$

The attempts to clarify the depletion charge effects have mostly been confined to very simple charge distributions as analysis of realistic distributions, even when depletion charge density is homogeneous, via analytical approaches can become a formidable problem. Only a few studies exist that try to analyze the effects of continuous depletion charge distributions on the observable properties in relatively thick films$^{7,38–41}$ but these studies have considered the single domain states. The possibility that, due to the inhomogeneous nature of the system owing to depletion charges, the transition could be into multidomain states even in structures with ideal electrodes would make a prominent difference in the calculated transition temperatures, which is one of the main emphases given in this paper. The way in which phase transition characteristics would be altered is discussed rigorously by Bratkovsky and Levanyuk$^{29}$ in the absence of dead layers. Reduction in the critical temperature commensurate with smaller coercivities in the ferroelectric state was demonstrated along with a qualitative discussion on the possibility of domain formation. One could easily foresee that the conclusions withdrawn for systems with ideal electrodes will have to be modified, for instance, for systems that have imperfect film-electrode interfaces, namely real electrodes. The real electrodes with finite Thomas-Fermi screening can be modeled as a thin dead layer between the electrode and the FE film, as demonstrated by Bratkovsky and Levanyuk in their 2009 paper (See Ref. 29). This latter statement is indeed a very important one when discussing experimental results on ferroelectric stability in the light of electrostatic considerations.

In this article, we address the question as to whether or not depletion charge effects could compete and overwhelm dead layer effects due to conditions at the film-electrode interfaces. To probe the competing energies, we use the Landau-Ginzburg-Devonshire (LGD) formalism for ferroelectric materials coupled with the interface conditions and presence of depletion charges. Firstly, films of various thicknesses with perfect film-electrode interfaces, namely ideal electrodes, but with depletion charges, are analyzed. A saw-tooth–type domain structure forms in relatively thick films due to the inhomogeneous internal field. At the transition temperature, thick films with ideal electrodes, but high depletion charge density always exhibit the saw-tooth domains. The period of this domain structure grows with increasing film thickness. Following this analysis, we introduce thin dead layers at the film-electrode interfaces to find out the possible alterations to the domain configurations and sensitivity of the domains to thickness effects. We found out that the domain period in a film having dead layers is altered upon introduction of a homogeneous depletion charge density to the system. At high depletion charge densities, domains with a saw-tooth type structure develop regardless of the presence of the dead layers. We also show that the transition temperatures are significantly lowered in relatively thick films with high depletion charge densities and dead layers, while this lowering is minimal in the thinner films and remain nearly unchanged with respect to charge-free films with dead layers. This behavior is a direct consequence of the dead layer effects dominating at lower thicknesses, while thicker films are under a heavier influence of depletion charges. Our results reveal the magnitudes of changes that can be expected in the transition temperatures for films with depletion charges, considering, especially, the transition into multidomain states.

II. THEORY AND METHODOLOGY

In this section, we give the governing equations and boundary conditions used to obtain field- and temperature-dependent characteristics of the ferroelectric thin film capacitors. The schematic of the geometry considered is given in Fig. 1. A two-dimensional grid is constructed that has $200n \times k$ cells, where $k(n)$ is the number of cells along the film thickness (width) and each cell has a dimension of 0.4 nm, nearly the lattice parameters of well-known pseudo-cubic perovskites, such as BaTiO$_3$ (BT), to imitate the order of lengths at which $P$ can vary in the system compared to real systems. Polarization is obtained by solving the equations of state derived from the LGD free energy for all $P$ in our system for an epitaxial monodomain (001) BT.
ferroelectric film on a (001) SrTiO$_3$ cubic substrate along with the Maxwell equation for dielectric displacement, employing a finite difference discretization. The strain states of the films determine the stable $P$ components. We partition the thin film capacitor system along the thickness axis, $z$, as follows:

$$w = 1 \quad \text{when} \quad -h/2 \leq z \leq +h/2, \quad w = 0 \quad \text{when} \quad -h/2 - d < z < -h/2 \quad \text{and} \quad +h/2 < z < d + h/2,$$

where $w$ is a step-wise function defining the interface between the dead layer and the ferroelectric, $d$ is the dead layer thickness (taken as 1-unit cell-thick, 0.4 nm in this work), and $|h|$ is the thickness of the ferroelectric layer. The electrode-dead layer interfaces are at $-h/2 - d$ and $d + h/2$, respectively. Note that $d = 0$ indicates the absence of a dead layer, i.e., a perfect film-electrode contact interface. The equations of state for the system to define the relation between the fields in the layers and the $P$ components as well as $\varepsilon_v$ of the dead layers using the definition of $w$ in Eq. (1) are

$$w \left[ 2x_{111}^0P_3 + 4x_{113}^0P_3P_1^2 + 4x_{133}^0P_3^2 + 6x_{111}^0P_5 + x_{112}(4P_3P_4^1 + 8P_3^2P_1^2) + 2x_{123}P_3P_4^1 - G \left( \frac{\partial^2 P_3}{\partial z^2} - \frac{\partial^2 P_3}{\partial x^2} \right) \right] + (1 - w) \frac{D_3}{\varepsilon_v e_0} = wE_3^F + (1 - w)E_3^d,$$

$$w \left[ 2x_{111}^0P_1 + (2x_{111}^0 + x_{112}^0)P_3 + 2x_{111}^0P_1^2 + 6x_{111}^0P_5^2 + 2x_{112}(3P_5 + 3P_3^2P_1P_4^1) + 2x_{123}P_3P_4^2 - G \left( \frac{\partial^2 P_1}{\partial z^2} + \frac{\partial^2 P_1}{\partial x^2} \right) \right] + (1 - w) \frac{D_1}{\varepsilon_v e_0} = wE_1^F + (1 - w)E_1^d,$$

in the dead layers, with $\phi^F$ and $\phi^d$ being the electrostatic potential in the ferroelectric and the dead layer, respectively. The electrostatic potential in each layer can be found at each point as a function of $P$ and the dielectric constant of the dead layer using the Maxwell relation in the absence of free charges $\nabla \cdot D = 0$ and $\nabla \cdot D = \rho$ when depletion charges due to ionized impurities are present. $\rho$ is the volumetric charge density (0 when no impurities are present). Thus, one has

$$\frac{\partial^2 \phi^F}{\partial z^2} + \frac{\partial^2 \phi^F}{\partial x^2} = \frac{1}{\varepsilon_v e_0} \left( \frac{\partial P_3}{\partial z} + \frac{\partial P_1}{\partial x} - \rho \right)$$

in the ferroelectric layer and

$$\frac{\partial^2 \phi^d}{\partial z^2} + \frac{\partial^2 \phi^d}{\partial x^2} = -\frac{\rho}{\varepsilon_v e_0}$$

for the dead layer. We assume that each impurity contributing to $\rho$ has only one positive unit charge (the charge of one electron) in all cases. The depletion charge density in this work, both in full and partial depletion cases, is assumed to be constant throughout the film volume, which is indeed realistic enough, especially for thicknesses at the order of a few tens of nanometers (See Refs. 30, 36, and 37), where full depletion is possible. The boundary conditions we employed for $P_{1,3}$ are

$$\left[ P_1 + \lambda \frac{\partial P_1}{\partial z} \right]_{z = \pm \frac{h}{2}} = 0, \quad \left[ P_3 + \lambda \frac{\partial P_3}{\partial x} \right]_{z = \pm \frac{h}{2}} = 0,$$
at the top and bottom electrode-film interface of the ferroelectric, where the extrapolation length, $\lambda$, is taken as infinite. Periodic boundary conditions are used along the $x$-axis, i.e.,

$$P_3(z, x = 0) = P_3(z, x = L), P_1(z, x = 0) = P_1(z, x = L). \quad (10)$$

We apply Dirichlet boundary conditions in the electrostatic equations to solve $P$ in the thin film capacitors. At the dead layer-electrode interfaces, $-h/2 - d$ and $h/2 + d$ ($d = 0$ corresponds to ideal electrodes), $\phi = 0$ corresponds to short-circuit boundary conditions between the electrodes. Figure 1 shows the geometry adopted. Note that the entire “capacitor system” is neutral as the charges from ionized impurities, whose density is $\rho$, accumulate on the electrodes, and, thus, the number of positive and negative charges are equal.

Equations of state [Eqs. (2a) and (2b)] along with the equations of electrostatics in [Eqs. (7) and (8)] using relations given in Eqs. (3) and (6) are solved simultaneously for $P$ components employing a Gauss-Seidel iterative scheme subject to boundary conditions mentioned above in Eqs. (9) and (10). The simulations always start with small fluctuations of $z$ and $x$ components of $P$ around zero that later on develop into the domain structure, depending on dead layer and film thickness. We limit ourselves to 10 000 iterations converging to a difference of about $10^{-8}$ between consecutive iterative $P$ solution steps when ferroelectricity exists. Owing to the compressive in-plane misfit in (001) BaTiO$_3$ on (001) SrTiO$_3$ (about 2.5%), only $P_3$ is the spontaneous polarization that, in addition to when depletion charge exists, also contains the built-in polarization, $P_b$. Thus, from here onwards, the ferroelectric part of $P_3$ will be denoted as $P_f$ and the built-in part as $P_b$. Note that, when $\rho = 0$, there is only one solution and it is $P_3 = P_f$.

III. RESULTS AND DISCUSSION

A. Room temperature domain structures when $d = 0$ (ideal electrodes)

We start discussing our results for three different film thicknesses, 12 nm, 16 nm, and 20 nm, with perfect film-electrode interfaces obtained at room temperature (RT), assuming that these films are fully depleted (Part of our results are also given in Ref. 44). This assumption can be justified by noting that, for instance, in the case of impurity densities around $10^{26}$, one might expect depletion zone thickness of around 8–10 nm, depending on the dielectric constant attaining values around 100–200 in a typical ferroelectric at room temperature, which indicates a total depletion zone of 16 to 20 nm for a film with top and bottom electrodes if the interfaces are symmetric. Structures with depletion charge at the max density limit considered in our work (2 $\times$ $10^{26}$ ionized impurities/m$^3$) that are thinner than 10-nm thickness are nearly always found to exist in an imprinted single domain state and are not of interest here. The reason for this outcome is discussed in the proceeding paragraphs. The films without any depletion charge also exist in a homogeneous monodomain state and are not discussed here again for brevity. In general, throughout this work, we chose to study two different depletion charge densities that reflect moderate-high and very high impurity densities reported for such structures. Depletion charge densities as high as $10^{25}$–$27$ ionized impurities/m$^3$ were reported, and we remain around these (5 $\times$ $10^{25}$ ionized impurities/m$^3$ for the moderate-high limit and 2 $\times$ $10^{26}$ for the high limit) values in our simulations.

Figure 2 displays the domain structures that form in films of various thicknesses that have a fixed volumetric depletion charge density corresponding to 2 $\times$ $10^{26}$ ionized impurities/m$^3$. Upon finding that low densities of depletion charge yield only a unidirectional $P_f$ in thin films, we focus on the densities that do trigger domains in thick structures (>10 nm). As a comparison, for example, the 8-nm-thick film with the aforementioned depletion charge density (2 $\times$ $10^{26}$ ionized impurities/m$^3$) does not undergo a domain stabilization, owing to the “insufficient extent of inhomogeneity”, meaning it is not thick enough for the built-in field to render a highly inhomogeneous structure considering full depletion. By inhomogeneous, we mean here the dependence of the local Curie temperature on the built-in electric field at that location. Extremely high densities of depletion charge (>10$^2$ ionized impurities/m$^3$) could perhaps stabilize domains in thicker films (>28 nm) with ideal electrodes, but are out of the main scope of our study.

In Fig. 2, we give the total polarization along with the ferroelectric polarization, and the latter is obtained by subtracting $P_3$ from $P_f$ for 12 nm, 16 nm, and 20 nm films. The $P_b$ is found by running our calculations above the Curie point, as it is nearly temperature insensitive and is the only corresponding solution satisfying Eqs. (2a), (7), and (8) [please see Fig. 3(a) for $P_b$ when full depletion is assumed for a 20-nm film]. At 2 $\times$ $10^{26}$ ionized impurities/m$^3$, a sawtooth--type domain pattern develops at RT, whose period is a function of thickness. Relatively lower depletion charge densities (<$10^{26}$ ionized impurities/m$^3$) do not tend to stabilize domains and result in a uniaxial $P_f$ whose amplitude is less in one half of the film than in the other half, concomitantly
with the internal field distribution. Therefore, the formation of domains in thicker films is due to the highly inhomogeneous nature of the built-in field renormalizing the linear term in $P_3$ in Eq. (2a). Here, the amplitude of the variation in the local transition temperature naturally becomes more profound toward the film boundaries, with increasing thickness for a given constant charge density. The situation described that applies to our analysis is also schematically depicted in Figs. 3(a)–3(c) for clarity. Hence, thicker structures are forced to undergo domain stabilization to minimize the depolarizing fields emanating from the gradient of the polarization induced by the inhomogeneous built-in field. The domain period in such a high inhomogeneous system becomes a function of position, somewhat similar to what has been reported for discrete, artificially graded structures, where the authors calculated domain fractions.

Those results reveal that the domain structures forming due to the depletion charge–induced fields in systems with ideal electrodes is quite different from what occurs when dead layers are present. For instance, in the latter, ferroelectric polarization amplitude attains a maximum in the middle section of the film, while saw-tooth–type domains have the maximum amplitude of the ferroelectric polarization wave near the electrode interfaces.

We find it important to add that, for depletion zone widths smaller, but comparable to film thickness where a charge-free region exists, we still see a similar behavior to a fully depleted film. $P_b$ in the case of partial depletion is provided in Figs. 3(a)–3(c) for a 20-nm film with 8 nm depletion at each interface. If one carries out a more complete depletion width, $w_d$, calculation within the assumption that the ionized impurities have a constant density in the depletion zone using

$$w_d = \left[ \frac{2\varepsilon_0 \varepsilon_r (\phi_d - \phi_0)}{N_d q} \right]^{1/2}$$

for a dielectric constant of the ferroelectric ($\varepsilon_r$) being at the order of 200 and for the number of ionized impurities around $2 \times 10^{25}/\text{m}^3$, one finds that depletion width starting from the electrode-film interface is about 8 nm in the film. In Eq. (11), $\phi_d$ and $\phi_0$ are the depletion potential (or barrier from the metal side) and external applied potential, respectively, (zero when discussing $w_d$ in this formulation). Domain structures far below the transition (room temperature) for the aforementioned partial depletion in the case of a 20-nm-thick film is provided in Fig. 4. For the full-depletion assumption, we had found that a saw-tooth–type domain configuration is stabilized. For a 20-nm-thick film, this would mean that there is a depleted region having a thickness of $2 \times 8$ nm = 16 nm at the interfaces, with a charge free region in the middle section. Our calculation shows that the saw-tooth domain structure is still present despite the charge-free region in the middle of the film, as given in Fig. 4 for 20-nm, 24-nm, and 28-nm-thick films. This behavior is quite straightforward to justify: The domains emanating at the interfaces where a strong built-in field exists do now want to terminate in the charge-free middle section with either a head-to-head or tail-to-tail domain configuration, which is energetically costly. Moreover, the domain walls in the charge-free middle section are perpendicular to the film plane to minimize the total energy. In films with thickness being equal to or smaller than $2 \times w_d$, the situation converges to full depletion, already analyzed in this work.

For sufficiently thick films with high impurity density, the depletion zone in the film will be confined to a relatively small volume near the electrodes and the entire film could be

\[ \text{FIG. 3. (Color online) Schematic of the built-in field plotted as a function of position along the thickness of the ferroelectric film for a } 2 \times 10^{25} \text{ impurity/m}^3 \text{ charge density in the case of (a) full depletion, (b) partial depletion, and (c) comparison of the built-in polarization along the thickness of full and partial depletion cases.} \]

\[ \text{FIG. 4. (Color online) Domain structures in the case of (a) 20-nm, (b) 24-nm, and (c) 28-nm-thick films for partial depletion consideration for a } 2 \times 10^{25} \text{ impurity/m}^3 \text{ charge density. The depleted layers are 8-nm-thick into the film starting from the film-electrode interfaces.} \]
free from saw-tooth-type domains existing in a single domain state. Moreover, in the case of low impurity density and thicker films \((2 \times w_d \approx t\), where \(t\) is film thickness), a domain-forming built-in field might not be expected, but the built-in field will shift the hysteresis along the field axis and smear the transition anomalies, as demonstrated in a previous work, as well as reducing the Curie temperature.

**B. Room temperature domain structures when** \(d = 1\) \(\text{(dead layers present)}\)

In the presence of dead layers \((d = 1\) unit cell\) and depletion charge, a competition between the two formations, each of which is a source of inhomogeneity, takes place. Here, we focus on fully depleted films with dead layers. A set of structures at RT for three different thicknesses and two depletion charge densities are provided in Fig. 5. The left-hand side gives the domain structure in the absence of depletion charge, while the right-hand side is when depletion charge is present. Subtracting the \(P_{b}\) at each site from \(P_{3}\), we again get the \(P_f\), as we did in Subsection III A. Among the analyzed structures, relatively moderate density of depletion charge \((5 \times 10^{25}\) ionized impurities/m\(^3\) in this work) slightly alters the domain wall angles with respect to the film normal along with a period change, as will be discussed next. A charge density of \(2 \times 10^{26}\) stabilizes the saw-tooth domain structure that has the prominent maxima in the \(P_f\) profile at the domain tips, similar to the case when \(d = 0\). Such a formation indicates that thick films with high depletion charge densities are under “weaker influence” of the dead layers. Another effective way to enable the comparison of the domain periods in films with and without depletion charge for a given thickness would be to plot and discuss the wave vector \(k\) of domains \((k = \frac{2\pi}{\lambda}, \text{where} \lambda \text{is domain period})\) as a function of thickness, as we do in the preceding paragraphs.

Before discussing the probable changes in domain period when depletion charges are present in thin films, we give the results for the domain wave vector, \(k\), we obtained both in our simulations and using the approach presented in Ref. 46 to validate the trends of our simulations for charge-free films in Fig. 6(a). A summary of the approach in Ref. 46 in a modified form (see also Ref. 47) is given in the Appendix for convenience. Our numerical results are in excellent agreement with the results obtained using the methodology in the Appendix, confirming the reliability of the method. Note the approach in the Appendix adopted from Refs. 46 and 47 analyzes the phase transition point, considering linear equation of state. We find that the domain period does not nearly change at all with further cooling upon the transition from the paraelectric to the multidomain FE state and transforms from a sinusoidal pattern to a square-like one, making it feasible to compare \(k\) values at and below the transition. In other words, even when our simulation temperatures are not the same as the

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**FIG. 5.** (Color online) Domain structures for (a) 12-nm, (b) 16-nm, and (c) 20-nm-thick films with dead layers. On the left-hand side are the domain structures for 0 impurity density and \(d = 1\). The right-hand side of each color map for a given thickness are the domain structures for impurity densities of \(5 \times 10^{25}\) (upper color map) and \(2 \times 10^{26}\) (lower color map). Scales are given to display the range of \(P_3\) in C/m\(^2\).
temperatures at which the $k$ values were found using the approaches in Refs. 46 and 47, the $k$’s in both their study and our simulations are directly comparable.

To visualize the impact of depletion charge on the domain structures in films with dead layers, we now discuss the behavior of the wave vector, $k$, of the $P_1$ wave plotted as a function of film thickness for ionized impurity densities of $5 \times 10^{25}$ and $2 \times 10^{26} \text{ m}^{-3}$. Our results for films at RT without and with depletion charge are in Fig. 6(b). The presence of electrical domains in films with depletion charge has persisted for the entire thickness range of interest in our study. Domain period for films thinner than 12 nm with $5 \times 10^{25}$ ionized impurities/m$^3$ is smaller than the charge-free film, while $2 \times 10^{26}$ ionized impurities/m$^3$ follows more or less the charge-free film, but with slightly larger $k$ values (i.e., smaller domain period). The general trend of the increase in $k$ values for films thinner than 12 nm in our work might be perceived as an indication that the depletion charge amplifies the depolarizing field for a given set of material parameters (domain wall energy, fixed dead layer thickness, dielectric constant, etc.). But this trend changes with increasing film thickness for the films having $5 \times 10^{25}$ ionized impurities/m$^3$ with respect to the charge-free case.

Around 15 nm, a crossover occurs, after which the thicker films with $5 \times 10^{25}$ ionized impurities/m$^3$ carrier density develop a coarser domain structure. Here, from the data of our simulations, we can see that the domain period is altered in a way the depolarizing field appears to be amplified, leading to a finer domain period, hence, a larger $k$.

Still, we cannot arrive at general conclusions for the entire thickness regime we considered, as thicker films (>16 nm) with moderate-high depletion charge density have a distinctly different domain period. Despite the thought that any formation giving rise to or amplifying depolarizing fields will reduce the transition temperature, comparing the $k$ values for a given thickness does not lead us to conclude so. To analyze the effect of depletion charge effects on the transition temperature, we carry out cooling runs in our simulations and extract and discuss the transition temperatures in Subsection III C.

C. Phase transition temperatures

This section considers the case of films where the entire volume is depleted, as we note that a further detailed discussion might be necessary to compare partial depletion and full depletion effects for various impurity densities. Furthermore, for asymmetrical film-electrode interfaces, it is clear that such effects might be altered significantly. We had already mentioned, for symmetrical film-electrode interfaces, that films with high impurity densities, where the depletion zones are confined to the near-electrode region, can exist in a single domain state and the transition temperatures will set accordingly. We leave this comparison for future work and focus on films with depletion zones being at the order of the film thickness, i.e., full depletion. The paraelectric-ferroelectric transition temperatures for films with full depletion are expected to be lowered in the presence of depletion charges, dead layers, or when both coexist. Film thickness importantly comes into play in all the cases above (See Figure 7). Here, we emphasize the situation when dead layers and depletion charges are both present, but also run a case where the films at a thickness range of 3.2 nm to 24 nm have ideal electrodes for comparison. For reference, we first computed the transition temperature as a function of film thickness for a fixed dead layer thickness ($d = 1$) and dielectric constant ($\varepsilon_r = 20$) and our results are in Fig. 7(a) along with the results we obtained using the method prescribed in the Appendix. We find the transition temperatures by tracking $\langle |P_s| \rangle$ in our simulations. The transition temperatures computed from the numerical solution of Eq. (16a) in the Appendix have a very good match with the simulation results presented in this work, again confirming the validity of the prescribed method in Sec. II (Figure 7(a)). It must be borne in mind that the approach of Ref. 45 excludes the gradient of $P_s$ (total polarization) along the thickness of the film, which we do include in our study. This can be the possible cause of the slight deviation between the two results at small thicknesses. As expected, decreasing film thickness results...
in a reduction of the transition temperature, with domain period subsequently becoming comparable or larger than the film thickness. Note that we do not go down to very low temperatures, where ultrathin films (<3.2 nm in this work) might be in a single domain state upon transition from the paraelectric phase and that this takes place at quite low temperatures.

Computing the phase transition temperatures in films with dead layers, but now with two different depletion charge densities, we note that films with a charge density of $5 \times 10^{25}$ ionized impurities/m$^3$ have nearly the same transition temperature compared to the charge free ones for a given thickness [see Fig. 7(b)]. Note that a homogeneous charge distribution does not lead to any net bias fields between the electrodes, and no smearing of the transition exists, meaning the transition temperature is sharp. We then carried out the cooling runs for films having a depletion charge density of $2 \times 10^{26}$ ionized impurities/m$^3$ both in the presence and absence of dead layers to detect the transition.

As mentioned previously, tracking $\langle |P_3| \rangle$ and comparing it with $\langle P_3 \rangle$ allows us to detect the phase transition, if it is into a multidomain state. These films with $2 \times 10^{26}$ ionized impurities/m$^3$ and dead layers have a similar trend with the charge-free films at small thickness, but then the transition temperature is significantly reduced for thicker films. Moreover, the transition temperatures in thicker films with and without dead layers are nearly the same. This scenario is certainly different for thinner films (<12 nm), and it is seen that the dead layers entirely dominate the transition characteristics [compare the curves for the films having $2 \times 10^{26}$ ionized impurities/m$^3$ with and without dead layers in Fig. 7(b)]. This is solely due to the “degree of induced inhomogeneity” in the thicker films, where the built-in electric field due to depletion charges induce a strong gradient of the transition temperature via normalization of $x_i^m$ in Eq. (2a), causing a larger amplitude variation of $P_3$, possibly overriding dead layer effects. Therefore, we provide quantitative evidence that the thicker films will be under a stronger influence of the depletion charge effects compared to thinner ones. One must remember here, however, that we discuss the case of rather high densities of depletion charge. For moderate-to-low densities ($<10^{25}$ ionized impurities/m$^3$ in this work), the above discussion on transition temperatures merely converges to discussion of dead layer effects on the transition temperature as a function of film thickness.

IV. CONCLUSIONS

We have analyzed the phase transition characteristics of ferroelectric thin films with and without depletion charge, considering ideal electrodes and film-electrode interface with dead layers. Using the non-linear Landau-Ginzburg-Devonshire equation of state, simulations were carried out for films with different thicknesses at different temperatures to find the domain periodicities and transition temperatures as a function of depletion charge density at various thicknesses. The approach adopted from Refs. 46 and 47 has been used as a guide to check the validity of our simulation results. (001)BaTiO$_3$ grown on (001)SrTiO$_3$ with pseudomorphic electrodes was used as an example system. Films with high depletion charges split into saw-tooth–type domains, even when ideal electrodes are present. This happens when the film is above a critical thickness, below which a single domain, imprinted state is stabilized. Increase in film thickness naturally creates larger variations in the electric field, hence, in local transition temperatures, due to a constant density of depletion charge, and a saw-tooth–type domain structure is favored, even in films with ideal electrodes to minimize the depolarizing fields. Partially depleted films, when the depleted volume is comparable to the film volume, could still develop saw-tooth-type domains. The latter happens with the exception that the domain walls are parallel to the film normal to reduce domain wall energy. Presence of dead layers when depletion charge densities are not very high ($<10^{26}$ ionized impurities/m$^3$) determine the transition temperature both for thin (<10 nm) and thick films (>10 nm). At charge densities not very high, domain periods are slightly altered, subsequent with tilted domain walls with...
In the absence of free charges, \( \nabla \mathbf{D} = 0 \) both in the FE and the dead layers. Writing these conditions in terms of the potential and polarization in the FE film, we get

\[
\frac{\partial^2 \Phi_F}{\partial z^2} + \frac{\varepsilon_1}{\varepsilon_0} \frac{\partial \Phi_F}{\partial z} = \frac{1}{\varepsilon_0 \varepsilon_0} \frac{\partial \mathbf{P}}{\partial z}, \tag{A4}
\]

where \( \varepsilon_1 \) is the dielectric constant of the FE along the plane of the film (calculated as approximately 40 from the simulations, and this value is used) and

\[
\varepsilon_0 \left( \varepsilon_1 \frac{\partial^2 \Phi_d}{\partial z^2} + \varepsilon_1 \frac{\partial^2 \Phi_d}{\partial x^2} \right) = 0 \tag{A5}
\]

for the dead layer. For convenience, it is assumed that the dead layer is isotropic and \( \varepsilon_1^d = \varepsilon_2^d \), with \( \varepsilon_1^d \) being the dielectric constant of the dead layer along the film plane. The linear equation of state of the FE that is obtained by minimization of the Landau-Ginzburg free energy with its lowest order terms is

\[
AP - \frac{\partial^2 P}{\partial z^2} = -\frac{\partial \Phi_F}{\partial z}, \tag{A6}
\]

where the gradient of \( P \) along \( z \) has been neglected, as mentioned above, \( A = (T - T_c)/\varepsilon_0 C + M \), where \( T \) is temperature, \( T_c \) is the transition temperature in bulk form, \( C \) is the Curie constant, \( M \) represents any contribution of strain in the dead layer effects in thin films. While transition temperature \( T_h \) esystem analyze in Ref. 47, and we follow this general approach by full treatment of the film with high- \( k \) dead layers.

Going back to the system in Fig. 1, for a given dead layer thickness, \( d \) (1 BaTiO\(_3\) unit cell thick, \( \sim 0.4 \) nm, in this work), the boundary conditions of the system can be written as

\[
D_F^i - D_\xi^d = 0 \quad @ \pm L/2, \tag{A1}
\]

where \( D_F^i = \varepsilon_0 \varepsilon_0 E_F^i + P \) and \( D_\xi^d = \varepsilon_0 \varepsilon_0 E_\xi^d \) are the dielectric displacements in the FE and dead layers, respectively, with \( \varepsilon_0 \) being the permittivity of vacuum in International System of units (SI) units, \( \varepsilon_0^d \) is the dielectric constant of the dead layer, \( \varepsilon_0 \) is the background dielectric constant of the FE. \( P \) is the ferroelectric polarization in the FE along the film thickness, and \( L \) is the FE film thickness (see Fig. 1). The boundary conditions for the potential are as follows:

\[
\Phi_F = \Phi_\xi, \quad 0 \leq z \leq L/2, \tag{A2a}
\]

\[
\Phi_\xi = 0, \quad z = L/2 + d, \tag{A2b}
\]

\[
\Phi_\xi = 0, \quad z = -L/2 - d, \tag{A2c}
\]

where \( \Phi_F, \Phi_\xi \) are the potentials in the FE and the dead layer, respectively. The electric fields in the layers can then be found from the gradient of the potentials. From Eq. (A1), one gets

\[
-\varepsilon_0 \varepsilon_0 \frac{\partial \Phi_F}{\partial z} + P - \varepsilon_0 \varepsilon_0 \frac{\partial \Phi_\xi}{\partial z} = 0 \quad @ z = L/2. \tag{A3}
\]
where $A$, $B$, and $C$ are the amplitudes in the general solution. Using the BCs given in Eqs. (A1) and (A2c), we get two equations with two unknowns, $B$ and $C$ from Eqs. (A10) and (A11),

$$
B \left[ q \cos \frac{qL}{2} + \frac{q}{e_0\varepsilon g (A+g^2)} \cos \frac{qL}{2} \right] - e_0^2 k C \cosh - kd = 0,
$$

(A12a)

$$
B \sin \frac{qL}{2} - C \sinh - kd = 0.
$$

(A12b)

For a non-trivial solution to exist, the determinants of the coefficients in Eqs. (A12a) and (A12b) have to be zero, giving us

$$
B \left[ \sinh \frac{kd}{2} \left( q \cos \frac{qL}{2} + \frac{q}{e_0\varepsilon g (A+g^2)} \cos \frac{qL}{2} \right) \right] + e_0^2 k \cosh - kd \sin \frac{qL}{2} = 0,
$$

(A13)

meaning that

$$
\sinh \frac{kd}{2} \left( q \cos \frac{qL}{2} + \frac{q}{e_0\varepsilon g (A+g^2)} \cos \frac{qL}{2} \right) + e_0^2 k \cosh - kd \frac{qL}{2} = 0.
$$

(A14)

After some algebra on Eq. (A14), one gets

$$
\tan \frac{qL}{2} = \frac{\sqrt{|a_k|} \sinh \frac{kd}{2}}{e_0^2},
$$

(A15a)

where

$$
|a_k| = \left( \frac{1}{e_0 (A+g^2)} + 1 \right),
$$

(A15b)

which was previously obtained by the authors of Ref. 46 through a similar route. Their approach is somewhat repeated here for tractability in our paper. We solve Eq. (15) using a numerical approach and seek the $k$ value that yields the highest transition temperature from the paraelectric state into the ferroelectric state for a given $d$ (1-unit-cell thick in this work). We do not carry out the calculations in the single domain state regime, which correspond to thicknesses smaller than 3 nm and are outside the scope of our analysis. Also note that the described method is applied for the validation of the simulation results and do not reflect any depletion charge-related effects, which are separately given only by the numerical simulation presented in this paper.