The Determination of Implicit Polynomial Canonical Curves

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Abstract—A new method is presented for identifying and comparing closed, bounded, free-form curves that are defined by even implicit polynomial (IP) equations in the Cartesian coordinates $x$ and $y$. The method provides a new expression for an IP involving a product of conic factors with unique conic factor centers. The critical points for an IP curve also are defined. The conic factor centers and the critical points are shown to be useful related points that directly map to one another under affine transformations. In particular, the explicit determination of such points implies both a canonical form for the curves and the transformation matrix which relates affine equivalent curves.

Index Terms—Implicit polynomials, affine transformations, object recognition, pose estimation, canonical curves.

1 INTRODUCTION

A basic problem in computer vision is the automatic recognition of various 2D curves under affine transformations, and the subsequent determination of the affine transformation matrix which defines the pose of one curve relative to another (affine equivalent) curve. Several approaches have been employed to resolve this problem in the case of Euclidean transformations. The well-known scatter matrix approach [1] employs the eigenvectors of a matrix computed from a set of data points to determine the appropriate transformation. An alternative approach [4] uses least-squares comparisons based on data point correspondences for the images. A more recent implicit polynomial (IP) procedure [8] develops a closed-form solution for algebraic pose estimation which uses only the IP coefficients of the model and the test curve. However, the final function which estimates the rotation angle is a very involved nonlinear function of the coefficients.

Much recent work has focused on describing the curved outline of 2D objects by implicit polynomials of varying degrees [12], [5], [13], [7] and then identifying the objects using either algebraic or geometric invariants [12], [10], [6]. Affine invariants have also been used to recognize 3D objects bounded by smooth curved surfaces [15].

The pioneering work in [12] outlines some efficient techniques for computing algebraic invariants for both 2D curves and 3D surfaces from the eigenvalues of certain matrices constructed from the IP coefficients. When these invariants match those of another curve or surface, a (canonical) “intrinsic reference frame” is determined, and the coefficients of the defining polynomials can then be compared in this new coordinate system. The transformation matrices which relate those canonical curves which have similar coefficients can be determined from the transformations that define the curves.

This paper will develop some new procedures for solving the curve recognition and pose estimation problems, also from the IP coefficients of the curves. These procedures provide a new decomposition for an implicit polynomial involving a product of conic factors with unique conic factor centers. The critical points of an IP curve also are defined. The conic factor centers and the critical points are shown to be useful related points that directly map to one another under affine transformations, similar to the affine centroids and the Euclidean centers defined in [12], and the interest points defined in [10]. The explicit determination of such points directly implies a canonical form for the curves, with minimal computational expense, thus eliminating the need to compute and compare algebraic invariants. The transformation matrix which relates affine equivalent curves also is readily approximated, as we will illustrate.

The results presented in this paper can be used in numerous applications, such as indexing into databases where objects can be segmented from the background, for printed character and font recognition, for curve construction and alteration, and in certain vision-based control applications, to approximate the translational and rotational velocities of moving objects [16].

Section 2 formally introduces the implicit polynomial equations used here to represent planar curves. A unique conic factor decomposition for these IP equations is then given in Section 3. Section 4 establishes that the conic factor centers are related points under affine transformations, and Section 5 defines other related points, including all critical points. Section 6 then develops and defines our canonical curves and proves that affine equivalent curves have the same canonical curve. Section 7 presents some experimental results which illustrate our procedures for both the Euclidean and the affine cases, and Section 8 presents some concluding remarks.
2 IMPLICIT POLYNOMIAL CURVES

An algebraic curve of degree $n$ can be defined in the Cartesian \([x, y]\)-plane by the implicit polynomial equation:

\[
 f_n(x, y) = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \cdots + a_{n-1,1}x^{n-1}y + \cdots + a_{0n}y^n = \sum_{j=0}^{n} h_j(x, y) = 0 \tag{2.1}
\]

where each binary form \(h_j(x, y)\) is a homogeneous polynomial of degree \(r\) in the variables \(x\) and \(y\).

The number of terms in each \(h_j(x, y)\) is \(r + 1\), so that the IP equation defined by (2.1) has one constant term, two terms of the first degree, three terms of the second degree, etc., up to and including \(n + 1\) terms of the (highest) \(n\)th degree, for a total of \((n + 1)(n + 2)/2\) coefficients. Since an IP equation can be multiplied by any nonzero scalar without changing its zero set, an algebraic curve defined by \(f_n(x, y) = 0\) has \((n + 1)(n + 2)/2 - 1 = n(n + 3)/2\) independent coefficients. A monic polynomial \(f_n(x, y) = 0\) will be defined by the condition that \(a_{00} = 1\) in (2.1).

It will often be convenient to express a curve defined by (2.1) by a left-to-right ordered set of its coefficients, giving priority to the highest-degree forms first, and then proceeding to the lower-degree \(x\) terms next. In light of (2.1), such an ordered set of coefficients would be defined by the row vector

\[
 \begin{bmatrix}
 a_{00} & a_{n-1,1} & \ldots & a_{0,n} & a_{n-1,0} & a_{n-2,1} & \ldots & a_{0,n-1} & a_{n-2,0} & \ldots & a_{0,1} & a_{00}
 \end{bmatrix} \Leftrightarrow f_n(x, y) \tag{2.2}
\]

We finally note that odd curves are unbounded [13]. Therefore, to represent closed and bounded 2D curves, IP equations of even degree must be used. We will therefore restrict our attention to such curves in our subsequent discussions here, noting that the results can be extended to the more general cases [14], [16].

3 CONIC FACTORS, CENTERS, AND FACTORIZATIONS

The substitution of \(mx\) for \(y\) in any homogeneous form \(h_i(x, y)\) implies that

\[
 h_i(x, y = mx) = x' a_{0i}(m - m_1)(m - m_2) \cdots (m - m_r) H_i(m),
\]

or that

\[
 h_i(x, y = a_{0i}(y - m_1)(y - m_2) \cdots (y - m_r).
\]

Therefore, for any curve defined by (2.1), the highest degree form or the leading term, \(h_1(x, y)\), can be factored as above. Any complex roots of \(H_1(m) = 0\) will occur in conjugate pairs, so that the product of any conjugate pair of complex roots will imply a unique monic second degree term \(x^2 + q_1xy + r_1y^2\).

Since \(n/2 = p\) denotes the number of pairs of complex conjugate roots of the leading term of a bounded curve defined by (2.1), \(h_1(x, y)\) can be uniquely factored as the product

\[
 h_1(x, y) = \left( x^2 + q_1xy + r_1y^2 \right) \left( x^2 + q_2xy + r_2y^2 \right) \cdots Q_1(x, y) \quad \cdots \quad Q_p(x, y)
\]

where \(Q_j(x, y) = \prod_{i=1, i\neq j}^{2p} \left( x - q_{ij}y \right) \) for \(j = 1, \ldots, p\).

If \(Q_i(x, y) \neq Q_j(x, y)\) when \(i \neq j\), a total of \(p\) unique linear polynomials \(L_i(x, y) = s_i x + t_i y\) can then be determined such that

\[
 h_{n-1}(x, y) = L_1(x, y)Q_2(x, y)Q_3(x, y) \cdots Q_p(x, y) + \cdots + L_p(x, y)Q_1(x, y)Q_2(x, y) \cdots Q_{p-1}(x, y) \tag{3.2}
\]

Equations (3.1) and (3.2) then imply that the product

\[
 \prod_{i=1}^{p} Q_i(x, y) = h_n(x, y) + h_{n-1}(x, y) + r_{n-2}(x, y),
\]

for some “remainder” polynomial \(r_{n-2}(x, y)\) of degree \(n - 2\), which implies a unique conic factorization for \(f_n(x, y)\) defined by

\[
 f_n(x, y) = \prod_{i=1}^{p} Q_i(x, y) + f_{n-2}(x, y), \tag{3.3}
\]

where the \(n - 2\) degree polynomial

\[
 f_{n-2}(x, y) = \sum_{i=0}^{n-2} h_i(x, y) - r_{n-2}(x, y).
\]

If \(4r_1 \neq q_1^2\), each conic factor

\[
 C_i(x, y) = x^2 + q_1xy + r_1y^2 + s_ix + t_iy
\]

of (3.3), whose zero set passes through the origin because of its zero constant term, will have a unique conic factor center \(c_i \equiv (x_0, y_0)\) defined by the simultaneous solution of the linear equations [11]

\[
 \begin{bmatrix}
 2x + q_1y + s_i = 0 & \frac{dy}{dx} = q_1x + 2ry + t_i = 0
 \end{bmatrix}
\]

or by the matrix/vector relation

\[
 \begin{bmatrix}
 1 & q_1 & 2 & s_i & 2 & t_i \\
 q_1 & 2 & r_1 & q_1 & 2 & t_i \\
 0 & 0 & 0 & 0 & 0 & 0
 \end{bmatrix}
\]

Note that adding a constant term to \(C_i(x, y)\) doesn’t change its center.

1. A conic \(x^2 + q_1xy + r_1y^2 + s_ix + t_iy + r_1 = 0\) with \(4r_1 = q_1^2\) will plot as either a parabola or a pair of parallel lines [11], neither of which has a center.
4 AFFINE EQUIVALENCE AND RELATED POINTS

It is of obvious interest to determine when two algebraic curves are equivalent under an affine transformation \( A \), which is defined by both a linear transformation \( M \) and a translation \( P \); i.e.,

\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix} = \begin{bmatrix}
  m_1 & m_2 \\
  m_3 & m_4
\end{bmatrix} \begin{bmatrix}
  x' \\
  y'
\end{bmatrix} + \begin{bmatrix}
  p_x \\
  p_y
\end{bmatrix}
\]

(4.1)

In particular, any two \((n\text{th degree})\) curves defined by a monic \( f_a(x, y) = 0 \) and a monic \( \tilde{f}_a(x, y) = 0 \) will be affine equivalent if for some scalar \( s \),

\[
\begin{align*}
  f_a(x, y) &= 0 \\
  &\overset{A}{\rightarrow} s\tilde{f}_a(x, y) = 0
\end{align*}
\]

(4.2)

Two corresponding related points\(^2\) of \( f_a(x, y) \) and \( \tilde{f}_a(x, y) \), such as \( \{x_m, y_m\} \) and \( \{x_m', y_m'\} \), respectively, will be defined by the condition that

\[
\begin{bmatrix}
  x_m \\
  y_m
\end{bmatrix} = \begin{bmatrix}
  m_1 & m_2 & p_x \\
  m_3 & m_4 & p_y \\
  0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
  x_m' \\
  y_m'
\end{bmatrix}.
\]

Equation (4.2) clearly implies that two such related points satisfy the relation

\[
\begin{align*}
  f_a(x_m, y_m) &= z_m = s\tilde{f}_a(x_m, y_m) = s\tilde{z}_m \\
  &\Rightarrow s = \frac{z_m}{\tilde{z}_m}.
\end{align*}
\]

(4.3)

We next note that each conic \( C_i(x, y) \) in the factorization of \( f_a(x, y) \) can be written as

\[
C_i(x, y) = x^2 + q_i x y + r_i y^2 + s_i x + t_i y
\]

(4.4)

so that in light of (3.3),

\[
f_a(x, y) = \prod_{i=1}^{p} X^T M_i X + f_{i-2}(x, y) = 0.
\]

(4.5)

If (4.1) is substituted into (4.5), it follows that under an affine transformation \( A \),

\[
f_a(x, y) = 0 \overset{A}{\rightarrow} \tilde{f}_a(x, y) = 0
\]

\[
= \prod_{i=1}^{p} X^T M_i X + \tilde{f}_{i-2}(x, y) = 0
\]

for some polynomial \( \tilde{f}_{i-2}(x, y) \) of degree \( n - 2 \), where each symmetric

\[
M_i = A^T M_i A = \begin{bmatrix}
  \tilde{p}_i & \frac{\tilde{q}_i}{2} & \frac{\tilde{s}_i}{2} \\
  \frac{\tilde{q}_i}{2} & \frac{\tilde{r}_i}{2} & \frac{\tilde{t}_i}{2} \\
  \frac{\tilde{s}_i}{2} & \frac{\tilde{t}_i}{2} & \frac{\tilde{v}_i}{2}
\end{bmatrix}
\]

(4.6)

will define an affine transformed conic term

\[
X^T M_i X = \tilde{p}_i x^2 + \tilde{q}_i x y + \tilde{r}_i y^2 + \tilde{s}_i x + \tilde{t}_i y + \tilde{v}_i.
\]

(4.7)

Therefore, in light of (3.3) and (4.2), the affine transformed (monic)

\[
\tilde{f}_a(x, y) = \prod_{i=1}^{p} \tilde{C}_i(x, y) + \tilde{f}_{i-2}(x, y),
\]

where each monic conic factor

\[
\tilde{C}_i(x, y) = \prod_{i=1}^{p} \tilde{C}_i(x, y) + \tilde{f}_{i-2}(x, y),
\]

(4.8)

Equations (4.4) and (4.5) next imply that each conic term defined by (4.7), hence, each corresponding conic factor defined by (4.8) will have a unique conic factor center \( c_i = \{x_{ic}, y_{ic}\} \) defined by the matrix/vector relation

\[
\begin{bmatrix}
  \tilde{p}_i \\
  \tilde{q}_i \\
  \tilde{r}_i \\
  \tilde{t}_i \\
  \tilde{v}_i
\end{bmatrix} = \begin{bmatrix}
  \tilde{p}_i \\
  \frac{\tilde{q}_i}{2} \\
  \frac{\tilde{r}_i}{2} \\
  \frac{\tilde{t}_i}{2} \\
  \frac{\tilde{v}_i}{2}
\end{bmatrix} = \begin{bmatrix}
  x_{ic} \\
  y_{ic}
\end{bmatrix}.
\]

(4.9)

LEMMA 1. The conic factor centers of affine equivalent curves are related points.

PROOF. Equations (4.6) and (4.9) imply that

\[
\begin{bmatrix}
  x_{ic} \\
  y_{ic}
\end{bmatrix} = A^T M_i A \begin{bmatrix}
  x_{ic} \\
  y_{ic}
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0
\end{bmatrix}
\]

(4.10)

for \( K = \frac{s_i}{2} x_{ic} + \frac{t_i}{2} y_{ic} + y_{ic} \). Since the last column of \( (A^T)^{-1} \) is defined by \( \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \), (4.10) implies that

\[
M_i A \begin{bmatrix}
  x_{ic} \\
  y_{ic} \\
  1
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0 \\
  K
\end{bmatrix}
\]

(4.10)

so that in light of (3.5) and (4.4),

\[
A \begin{bmatrix}
  x_{ic} \\
  y_{ic} \\
  1
\end{bmatrix} = \begin{bmatrix}
  x_{ic} \\
  y_{ic}
\end{bmatrix}.
\]

5 OTHER RELATED POINTS

A critical-point (also called a stationary point [2]) of an IP function \( f_a(x, y) \) is defined as any point where both of its partial derivatives are zero; i.e., \( \frac{\partial f_a(x, y)}{\partial x} = 0 \) and \( \frac{\partial f_a(x, y)}{\partial y} = 0 \). The critical points for bivariate functions are minima, maxima, and saddle points.

\footnote{An acronym for real equivalent locations that affine transformations equate directly.}
LEMMA 2. The critical points of bivariate IP functions are related points.

PROOF. The chain rule of partial differentiation implies that

\[
\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \quad \Rightarrow \quad \frac{\partial f}{\partial x} = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \frac{\partial f}{\partial y}.
\]

Since \( f_n(x, y) = s f_n(x, y) \), in light of (4.2),

\[
\begin{bmatrix} \frac{\partial f_n(x, y)}{\partial x} \\ \frac{\partial f_n(x, y)}{\partial y} \end{bmatrix} = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix} \begin{bmatrix} \frac{\partial f_n(x, y)}{\partial x} \\ \frac{\partial f_n(x, y)}{\partial y} \end{bmatrix}.
\]

Therefore,

\[
\frac{\partial f_n(x, y)}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f_n(x, y)}{\partial y} = 0 \quad \Leftrightarrow \quad \frac{\partial f_n(x, y)}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f_n(x, y)}{\partial y} = 0.
\]

To establish the correct correspondence between the points in two sets of \( k \) corresponding related-points, such as \((x_i, y_i)\) and \((\overline{x}_i, \overline{y}_i)\), we next note that if \( f_n(x_i, y_i) = z_i \) and \( f_n(\overline{x}_i, \overline{y}_i) = \overline{z}_i \), as in (4.3), then \( z_i = s \overline{z}_i \), and

\[
s = \frac{z_i}{\overline{z}_i} = \frac{\sum_{i=1}^{k} z_i}{\sum_{i=1}^{k} \overline{z}_i}.
\]

Therefore, we will always “order” the related-points so that \( z_1 < z_2 < \ldots < z_k \), and

\[
z_1 < z_2 < \ldots < z_k \text{ if } s > 0, \quad \text{and} \quad \overline{z}_1 > \overline{z}_2 \ldots > \overline{z}_k \text{ if } s < 0.
\]

In the case of affine transformations, bilangent points, inflection points, and centroids also are related points which can be determined from knowledge of the curves. It might be noted that bitangent points and centroids are quite difficult to determine from knowledge of an IP equation. Inflection points, which correspond to the intersections of a curve with its Hessian, can be determined accurately, but with some computational effort. The critical points and the conic factor centers can be determined more easily, as we will now illustrate.

EXAMPLE 5.3. In light of (2.2), consider the quartic IP \( f_4(x, y) \) defined by the row vector

\[
[1, -2, 4, 4, 2, -3.03, 5.29, -3.69, -2.55, -0.002, -1.66, -4.53, 11.36, -2.22, -11.31].
\]

3. The fact that \( s = z_i / \overline{z}_i \) for all related-points, implies that for any two, \( z_i / \overline{z}_i = z_k / \overline{z}_k \). As a consequence, the related-point ratios \( z_i / \overline{z}_i = \overline{z}_j / \overline{z}_k \) will be absolute affine invariants that can be used to identify and compare IP curves [16].

Using the MATLAB [9] function “fmins,” the two (in this case) local minima of \( f_4(x, y) \) are found. The global minimum point is \([x_m, y_m] = (-0.387, 1.891)\), where \( f_4(-0.387, 1.891) = z_m = -27.861 \). This is shown in Fig. 1, which is a 3D plot of \( f_4(x, y) \) for all \( x, y \) points interior to the 2D curve defined by \( f_4(x, y) = 0 \).

Under the affine transformation matrix

\[
A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix},
\]

\( f_4(x, y) = 0 \) is mapped to an affine equivalent (monic) IP \( \tilde{f}_4(\overline{x}, \overline{y}) = 0 \) defined by the row vector

\[
[1, 1.619, 0.952, 0.238, 0.024, -0.529, 0.274, 1.4, 1.461, 0.561].
\]

The global minimum point of \( \tilde{f}_4(\overline{x}, \overline{y}) \) is at \( [\overline{x}_m, \overline{y}_m] = \{0.109, -1.605\} \), where

\[
\tilde{f}_4(\overline{x}_m, \overline{y}_m) = \overline{z}_m = -0.663,
\]

as shown in Fig. 2, which implies that

\[
s = \frac{z_m}{\overline{z}_m} = 27.861 / -0.663 = 42
\]
in light of (4.3) and (5.1).

Equation (3.3) next implies the conic factorization

\[
f_4(x, y) = C_1(x, y)C_2(x, y) + f_2(x, y),
\]

with

\[
C_1(x, y) = x^2 + 0.824xy + 0.334y^2 - 1.314x - 0.627y,
\]

\[
C_2(x, y) = x^2 - 2.824xy + 5.995y^2 - 1.717x + 3.622y,
\]

and

\[
f_2(x, y) = -2.257x^2 + 2.022xy - 2.259y^2 + 11.36x - 2.22y - 11.31.
\]

4. The authors have written numerous MATLAB m-files to compute the various functions presented in this paper. Certain of these m-files can be obtained by contacting either author at his e-mail address.
In light of (5.2), the conic factors \( C_1(x, y) \) and \( C_2(x, y) \) are so ordered because \( s = 42 > 0 \) and their centers

\[
c_1 = \{x_{1c}, y_{1c}\} = \{0.549, 0.261\}
\]

and

\[
c_2 = \{x_{2c}, y_{2c}\} = \{0.647, -0.15\}
\]

satisfy the relation

\[
z_1 = f_4(c_1) = -6.333 < z_2 = f_4(c_2) = -4.482.
\]

The affine equivalent IP \( f_4(x, y) = 0 \) implies a corresponding conic factor decomposition

\[
f_4(x, y) = C_1(x, y)C_2(x, y) + f_2(x, y),
\]

with

\[
C_1(x, y) = x^2 + 1.183xy + 0.373y^2 + 1.169x + 0.87y,
\]

\[
C_2(x, y) = x^2 + 0.436xy + 0.064y^2 - 2.27x - 0.343y,
\]

and

\[
f_2(x, y) = 1.752x^2 + 1.847xy + 0.572y^2 + 1.399x + 1.461y + 0.561
\]

so ordered by (5.2) because the corresponding conic factor centers of \( f_4(x, y) \), namely,

\[
\bar{c}_1 = \{x_{1c}, y_{1c}\} = \{1.739, -3.929\}
\]

and

\[
\bar{c}_2 = \{x_{2c}, y_{2c}\} = \{2.15, -4.653\}
\]

satisfy the relation

\[
\bar{z}_1 = \bar{f}_4(\bar{c}_1) = -0.1508 (= -6.333 / 42)
\]

\[
< \bar{z}_2 = \bar{f}_4(\bar{c}_2) = -0.1067 (= -4.482 / 42)
\]

Fig. 3 and Fig. 4 display the zero sets of both of the affine equivalent curves and their conic factors.

6 CANONICAL CURVES

If \( f_n(x, y) = 0 \) and \( \bar{f}_n(x, y) = 0 \) are affine equivalent IP curves, the known mappings of any three related-points of \( f_n(x, y) = 0 \) to any three corresponding related-points of \( \bar{f}_n(x, y) = 0 \), such as

\[
\{x_i, y_i\} \xrightarrow{A} \{\bar{x}_i, \bar{y}_i\} \quad \text{for } i = 1, 2, \text{ and } 3,
\]

will define the affine transformation matrix \( A \) via the relation

\[
\begin{bmatrix}
x_1 & x_2 & x_3 \\
y_1 & y_2 & y_3
\end{bmatrix}
\begin{bmatrix}
x_1 & x_2 & x_3 \\
y_1 & y_2 & y_3
\end{bmatrix}
\begin{bmatrix}
1 & T & \bar{T}
\end{bmatrix}
\Rightarrow A = T\bar{T}^{-1}.
\]

We next note that any three such related points of \( f_n(x, y) = 0 \) will define a canonical transformation matrix.
THEOREM 1. The implicit polynomial curves \( f_n(x, y) = 0 \) and \( \tilde{f}_n(x, y) = 0 \) will be affine equivalent \( \Rightarrow \) their canonical curves
\( f_n^*(x, y) = 0 \) and \( \tilde{f}_n^*(x, y) = 0 \) are the same, in which case the affine transformation matrix
\[
A = TE^{-1}A_c^{-1} = A_c A^{-1}.
\]

PROOF. \( \Rightarrow \) The affine equivalence of \( f_n(x, y) = 0 \) and \( \tilde{f}_n(x, y) = 0 \) implies (4.1), (4.2), and (6.1). The inverse of (6.3) and (6.4) then implies that
\[
s_n f_n^*(x, y) \xrightarrow{A^{-1}} f_n(x, y) \xrightarrow{A = A_c A^{-1}} \frac{A_c}{s_c} \tilde{f}_n^*(x, y)
\]
or that \( s_n f_n^*(x, y) = s_c \tilde{f}_n^*(x, y) \), with \( s_c = s_c \), since \( f_n^*(x, y) \) and \( \tilde{f}_n^*(x, y) \) are monic polynomials.
\( \Leftarrow \) Equation (6.3) and the inverse of (6.4) imply that
\[
\frac{f_n(x, y)}{s_c} \xrightarrow{A_c^{-1} A_c^{-1}} \frac{f_n(x, y)}{s_c} \xrightarrow{A = A_c A^{-1}} \frac{A_c}{s_c} \tilde{f}_n^*(x, y)
\]
or that
\[
\frac{f_n(x, y)}{s_c} \xrightarrow{A_c^{-1} A_c^{-1} A_c A^{-1}} \frac{A_c}{s_c} \tilde{f}_n^*(x, y).
\]

If the minimum-points of \( f_n(x, y) \) and \( \tilde{f}_n(x, y) \) determined in Example 5.3, namely,
\[
\{x_{n}, y_{n}\} = \{-0.387, 1.891\}
\]
and
\[
\{\tilde{x}_n, \tilde{y}_n\} = \{0.109, -1.605\}
\]
are used as a third pair of related-points to define \( T \) and \( \tilde{T} \), the corresponding canonical transformation matrices
\[
A_c = \begin{bmatrix} 0.549 & 0.647 & -0.387 \\ 0.261 & -0.15 & 1.891 \\ 0 & 0 & 1 \end{bmatrix}
\]
and
\[
\tilde{A}_c = \begin{bmatrix} 1.739 & 2.15 & 0.109 \\ -3.929 & -4.653 & -1.605 \\ 0 & 0 & 1 \end{bmatrix}
\]
will imply identical (in this ideal case) monic canonical curves via (6.3) and (6.4), namely, \( f_n^*(x, y) = 0 \) and \( \tilde{f}_n^*(x, y) = 0 \), both defined by the same row vector
\[
[1.0, 4.822, 7.085, 2.16, -4.462, -16.124, -19.502, -7.9, 5.485, 13.156, 7.933, -0.003, -0.003, -2.615],
\]
which verifies the affine equivalence of the two curves in light of Theorem 1.

Equation (6.1) finally verifies that the affine transformation matrix
\[
A = \begin{bmatrix} 0.549 & 0.647 & -0.387 \\ 0.261 & -0.15 & 1.891 \\ 0 & 0 & 1 \end{bmatrix}^{-1}
\]

\[
= \begin{bmatrix} 1.739 & 2.15 & 0.109 \\ -3.929 & -4.653 & -1.605 \\ 0 & 0 & 1 \end{bmatrix}
\]
and
\[
\tilde{A}_c = \begin{bmatrix} 1.739 & 2.15 & 0.109 \\ -3.929 & -4.653 & -1.605 \\ 0 & 0 & 1 \end{bmatrix}
\]
will imply identical (in this ideal case) monic canonical curves via (6.3) and (6.4), namely, \( f_n^*(x, y) = 0 \) and \( \tilde{f}_n^*(x, y) = 0 \), both defined by the same row vector
\[
[1.0, 4.822, 7.085, 2.16, -4.462, -16.124, -19.502, -7.9, 5.485, 13.156, 7.933, -0.003, -0.003, -2.615],
\]
which verifies the affine equivalence of the two curves in light of Theorem 1.

7 EXPERIMENTAL RESULTS

7.1 Fitting IP Models to Data

To apply our previous results, we will first fit IP curves to data sets. The IP fitting algorithm used here [7] is explicit least-squares, repeatable, Euclidean invariant, and numerically stable when compared to classical least-squares procedures.

Fig. 5 depicts two data sets which represent the outline of an airplane in two different Euclidean equivalent con-
...
which is Euclidean in this case, transforms the (lower outline) data set defined by \( \bar{x} \) and \( \bar{y} \) to the (upper outline) data set defined by \( x \) and \( y \).

The corresponding quartic IP \( f_3(x, y) = 0 \) is defined by the row vector
\[
\begin{bmatrix}
1, 1.622, 1.038, -59.896, 111.137, -0.0061, -5.737, -19.026, 42.731, -2.301, 3.715, 14.17, 0.802, 3.79, -0.204
\end{bmatrix}
\]
with global minimum-point \( \{x_m, y_m\} = \{0.923, -0.193\} \).

The corresponding transformed quartic IP \( \tilde{f}_3(\bar{x}, \bar{y}) = 0 \) is defined by the row vector
\[
\begin{bmatrix}
1, -1.759, 1.049, -0.211, 0.0213, -9.928, 12.69, -4.592, 0.476, 36.632, -29.641, 5.012, -59.214, 22.589, 35.321
\end{bmatrix}
\]
with global minimum-point \( \{\bar{x}_m, \bar{y}_m\} = \{2.119, -1.323\} \).

Our conic factorization of \( f_3(x, y) = C_1(x, y)C_2(x, y) + f_3(x, y) \) next implies that
\[
C_1(x, y) = x^2 + 5.915xy + 21.183y^2 + 0.956x + 2.709y,
\]
and that
\[
C_2(x, y) = x^2 - 4.293xy + 5.247y^2 - 0.962x + 1.346y,
\]
with centers at
\[
c_1 = \{-0.492, 0.0047\}
\]
and
\[
c_2 = \{1.685, 0.561\},
\]
respectively, so ordered because \( f_3(c_1) = -0.914 < -0.29 = f_3(c_2) \).

An analogous conic factorization of
\[
\tilde{f}_3(\bar{x}, \bar{y}) = \tilde{C}_1(\bar{x}, \bar{y})\tilde{C}_2(\bar{x}, \bar{y}) + \tilde{f}_3(\bar{x}, \bar{y})
\]
implies that
\[
\tilde{C}_1(\bar{x}, \bar{y}) = \bar{x}^2 - 1.505\bar{y}\bar{x} + 0.634\bar{y}^2 - 6.148\bar{x} + 4.955\bar{y},
\]
and that
\[
\tilde{C}_2(\bar{x}, \bar{y}) = \bar{x}^2 - 0.253\bar{y}\bar{x} + 0.0366\bar{y}^2 - 3.779\bar{x} + 0.488\bar{y},
\]
with corresponding centers at
\[
\tilde{c}_1 = \{1.25, -2.424\}
\]
and
\[
\tilde{c}_2 = \{1.857, -0.26\},
\]
respectively, so ordered because \( s = z_m / \Sigma_m = 58.294 > 0 \) and \( \tilde{f}_3(\tilde{c}_1) = -0.0218 < -0.0061 = \tilde{f}_3(\tilde{c}_2) \).

If the conic factor centers define the first two columns of \( T \) and \( \tilde{T} \), and the minimum-points the third columns, as in Example 6.5, (6.1) directly implies that
\[
\begin{bmatrix}
-0.492 & 1.685 & 0.923 \\
0.0047 & 0.561 & -0.193 \\
-2.424 & -0.26 & -1.323
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]

The monic canonical curve \( f_3(x, y) = 0 \) obtained by the transformation \( A_c = TE \) is defined by the row vector
\[
\begin{bmatrix}
1.0, 2.967, 2.94, -0.021, 0.228, -1.922, -2.78, -0.11, -0.679, 0.967, 0.272, 0.461, -0.001, -0.001, -0.071
\end{bmatrix}
\]
and the canonical curve \( \tilde{f}_3(\bar{x}, \bar{y}) = 0 \) obtained by the transformation \( \tilde{A}_c = \tilde{T}E \) is defined by the row vector
\[
\begin{bmatrix}
1.0, 2.976, 3.016, 0.02, 0.217, -1.939, -2.89, -0.184, -0.64, 0.991, 0.361, 0.402, -0.001, -0.001, -0.08
\end{bmatrix}
\]

Fig. 6 depicts these two nearly identical canonical curves.

### 7.2 Normalized Data Sets

The previous section illustrates how the minimum-points and the conic factor centers of quartic IP curves can be used to approximate the transformation matrix which relates two sets of data points that are Euclidean equivalent. Since the fitting algorithm used is Euclidean invariant, data sets that may be affine equivalent will be normalized before it is applied. Our normalization procedure will employ a linear rescaling, known as whitening [3], so that after this rescaling, the data set will have zero mean and a covariance matrix equal to the identity.

Consider a set \( S \) of \( N \) data points \( Y_i = [x_i, y_i]^T \) which outline the boundary of a 2D curve. The center \( G \) and the covariance matrix \( \Sigma \) of \( S \) are defined by
\[
G = \frac{1}{N} \sum_{i=1}^{N} Y_i = \left[ \begin{array}{c} x_i \\ y_i \end{array} \right]
\]
and
\[
\Sigma = \frac{1}{N-1} \sum_{i=1}^{N} (Y_i - G)(Y_i - G)^T
\]
respectively. Since the covariance matrix \( \Sigma \) is symmetric, it can be diagonalized by an orthogonal matrix \( U \) composed of the eigenvectors of \( \Sigma \), so that
\[
\Lambda = U^T \Sigma U \Rightarrow \Sigma = U \Lambda U^T,
\]
where Λ is a diagonal matrix composed of the eigenvalues of Σ.

Now consider another data set \( \hat{S} \), obtained from \( S \) via the transformation

\[
\hat{Y}_i = \begin{bmatrix} \hat{x}_i \\ \hat{y}_i \end{bmatrix} = \Lambda^{1/2} Y_i - G.
\]

(7.2)

In light of (7.1), its center will be at

\[
\hat{G} = \frac{1}{N} \sum_{i=1}^{N} \Lambda^{-1/2} Y_i - G
\]

which implies that

\[
\hat{G} = \Lambda^{-1/2} G - \Lambda^{-1/2} Y_i G = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}
\]

the origin, and its covariance matrix

\[
\hat{\Sigma} = \frac{1}{N-1} \sum_{i=1}^{N} \hat{Y}_i \hat{Y}_i^T
\]

\[
= \Lambda^{-1/2} \Sigma G - \Lambda^{-1/2} Y_i G = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}
\]

the identity matrix, which defines \( \hat{S} \) as a normalized data set obtained from \( S \).

Now consider two \( N \)-point data sets \( S \) and \( \bar{S} \) which define the boundaries of two affine equivalent curves. If the data sets are related by an affine transformation matrix \( A \), (4.1) will imply that

\[
Y_i = M \bar{Y}_i + P \text{ for } i = 1, 2, \ldots, N,
\]

which implies that \( G = M \bar{G} + P \). As a consequence, \( Y_i - G = M(\bar{Y}_i - \bar{G}) \), so that

\[
\Sigma = \frac{1}{N-1} \sum_{i=1}^{N} (\bar{Y}_i - \bar{G})(\bar{Y}_i - \bar{G})^T = M \Sigma M^{-T}.
\]

Equation (7.2) next implies that the normalized data sets \( \hat{S} \) and \( \bar{\hat{S}} \), obtained from \( S \) and \( \bar{S} \), respectively, will satisfy the relations

\[
\hat{Y}_i = \Lambda^{1/2} U^T (Y_i - G)
\]

and

\[
\bar{\hat{Y}}_i = \Lambda^{1/2} \bar{U}^T (\bar{Y}_i - \bar{G}) = \Lambda^{1/2} \bar{U}^T M^{-1} (Y_i - G).
\]

The first of these two equations implies \( (Y_i - G) = U \Lambda^{1/2} \hat{Y}_i \).

If this relation is used in the second equation, it follows that

\[
\bar{\hat{Y}}_i = \Lambda^{1/2} \bar{U}^T M^{-1} U \Lambda^{1/2} \hat{Y}_i,
\]

(7.3)

so that the linear transformation matrix

\[
M = U \Lambda^{1/2} R^{-1} \Lambda^{1/2} \bar{U}^T.
\]

(7.4)

We finally note that \( R \) is orthogonal, since

\[
\Lambda^{1/2} \bar{U}^T M^{-1} U \Lambda^{1/2} \bar{U}^T = \frac{1}{\Sigma} \sum_{i=1}^{N} (Y_i - G)(Y_i - G)^T M^{-1} \bar{U} \bar{U}^T
\]

\[
= \frac{1}{\Sigma} \sum_{i=1}^{N} (Y_i - G)(Y_i - G)^T = I
\]

In summary, if two affine equivalent data sets are normalized, and our Euclidean invariant fitting algorithm is used to determine IP equations which fit the normalized data sets, any three corresponding related-points can then be used to define \( T \) and \( \bar{T} \), and to compare the resulting canonical curves defined by (6.2) to (6.4). If they are “equal,” the normalized IP curves are rotationally equivalent, and the original data sets are affine equivalent.

To approximate \( A \), we next employ (6.1) to determine the orthogonal transformation matrix \( R \) which relates the normalized IP curves. The matrix \( M \), which defines the linear part of the affine transformation, can then be determined using (7.4), and the translation vector \( P \) can be obtained by substituting the centers of the original data sets in (4.1); i.e.,

\[
P = \begin{bmatrix} p_x \\ p_y \end{bmatrix} = \begin{bmatrix} x_c \\ y_c \end{bmatrix} - M \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}.
\]

(7.5)

To illustrate this procedure, consider Fig. 7a and Fig. 7b, which depict two data sets, \( S \) and \( \bar{S} \), that outline the boundary of a military jet. These data sets are related by the affine transformation matrix

\[
A = \begin{bmatrix} -0.594 & -0.56 & 2.4 \\ 0.787 & -0.860 & 1.1 \\ 0 & 0 & 1 \end{bmatrix}
\]

Eqn. 7c and Fig. 7d depict the corresponding normalized data sets, \( \hat{S} \) and \( \bar{\hat{S}} \), as well as superimposed sixth-degree implicit polynomial fits (the bolder curves). In particular, \( f(x, y) = 0 \), a sixth-degree IP fit of \( \hat{S} \), is defined by the (monic) row vector
then implies that
\[
\mathbf{C}_1(x, y) = x^2 - 0.189xy + 0.07y^2 - 1.407x + 0.097y,
\]
\[
\mathbf{C}_2(x, y) = x^2 + 2.312xy + 1.705y^2 + 2.204x + 2.686y,
\]
and
\[
\mathbf{C}_3(x, y) = x^2 - 3.221xy + 4.001y^2 + 1.108x - 2.585y,
\]
with centers at
\[
\bar{z}_1 = \{0.732, 0.297\},
\]
\[
\bar{z}_2 = \{-0.886, -0.187\},
\]
and
\[
\bar{z}_3 = \{-0.969, 0.284\},
\]
respectively, so ordered because
\[
s = \sum_{i=1}^{3} f_6(c_i) / \sum_{i=1}^{3} \bar{f}_6(z_i) = 12.375 > 0
\]
and
\[
\varpi_i = \bar{f}_6(c_i) < -1.185 < \varpi_2 = f_6(c_2) = -0.417 < \varpi_3 = \bar{f}_6(c_3) = -0.345.
\]
Since there are three conic factor centers in this case, the minimum-points of the normalized curves need not be used to define \( R \) which, in light of (7.3), is given by the inverse of (6.1); i.e.,
\[
\mathbf{T}^{-1} = \begin{bmatrix}
0.732 & -0.886 & -0.096 \\
0.297 & -0.187 & 0.284 \\
-0.028 & 0.199 & 0.298
\end{bmatrix} = \begin{bmatrix}
0.798 & 0.864 & -0.01 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}^{-1}
\]
\[
= \begin{bmatrix}
-0.279 & 0.935 & 0 \\
0.994 & -0.427 & 0 \\
0 & 0 & 1
\end{bmatrix}
\Rightarrow \mathbf{R} = \begin{bmatrix}
-0.279 & -0.935 \\
0.994 & -0.427 \\
0.01 & 0.427
\end{bmatrix}
\]

The monic canonical curve \( f_6'(x, y) = 0 \) obtained by the transformation \( \mathbf{T} \mathbf{E} \) is defined by the row vector
\[
[1.0, -4.583, 8.378, -8.349, 5.729, -2.91, 0.753, 0.852, -1.636, 0.591, -0.271, -0.02, 0.273, -3.038, 12.641, -18.377, 13.479, -3.877, -3.25, 5.69, -0.717, -1.665, 2.245, -10.88, 4.569, 0.218, -0.245, -1.048]
\]
and the monic canonical curve \( \bar{f}_6'(x, y) = 0 \) obtained by the transformation \( \mathbf{T} \mathbf{E} \) is defined by the row vector
\[
[1.0, -5.076, 10.156, -10.679, 7.328, -3.757, 1.047, 1.173, -2.935, 1.976, -1.099, 0.505, 0.177, -2.819, 13.518, -21.317, 16.9, -5.337, -4.031, 8.666, -3.436, -1.278, 1.552, -11.949, 5.875, 0.472, -0.712, -1.088].
\]

Fig. 8 depicts these two nearly identical canonical curves.

The normalization matrices
\[
\mathbf{U} = \begin{bmatrix}
0.769 & -0.639 \\
0.639 & 0.769
\end{bmatrix}, \mathbf{L} = \begin{bmatrix}
0.227 & 0.0 \\
0.0 & 0.146
\end{bmatrix}
\]
and
\[
\mathbf{U} = \begin{bmatrix}
-0.827 & -0.563 \\
0.563 & -0.827
\end{bmatrix}
\]
Moreover, the centers of the critical points of the unique conic factor decomposition of implicit polynomial equations of even degree \( n \) can be used to determine the rotation matrix \( R \). An approximation to the \( M \) matrix of the affine transformation \( A \) then follows from (7.4). The translation vector \( P \) is then determined via (7.5), using the centers of the original data sets, as illustrated in Section 7.2. Clearly, an affine invariant fitting algorithm would eliminate the need to normalize data sets.

Although the results presented here are restricted to closed and bounded 2D curves, they can be extended to the more general case \([14, 16]\). We are also investigating similar decompositions in 3D by restricting the IP equations to those which imply a factorization analogous to that defined by (3.3).

Data sets can be normalized before fitting IP curves to determine when they are affine equivalent. In particular, a Euclidean invariant fitting is first applied to the normalized data sets to determine if the normalized IP equations are rotationally equivalent (by comparing their canonical curves). If they are, (6.1) can be used to determine the rotation matrix \( R \). An approximation to the \( M \) matrix of the affine transformation \( A \) then follows from (7.4). The translation vector \( P \) is then determined via (7.5), using the centers of the original data sets, as illustrated in Section 7.2. Clearly, an affine invariant fitting algorithm would eliminate the need to normalize data sets.

**REFERENCES**


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