One – Memory in Repeated Games*

Mehmet Barlo
Sabancı University

Guilherme Carmona
Universidade Nova de Lisboa

Hamid Sabourian
University of Cambridge

October 11, 2006

Abstract

We study the extent to which equilibrium payoffs of discounted repeated games can be obtained by 1 – memory strategies. To this end, we provide a complete characterization of the 1 – memory simple strategies and use it in games with 3 or more players each having a connected action space, to show that:

1. all subgame perfect payoffs can be approximately supported by a 1 – memory contemporaneous ε – perfect equilibrium strategy,
2. all subgame perfect payoffs satisfying Abreu (1988) type of incentive conditions strictly, can be approximately supported by a 1 – memory subgame perfect strategy, and
3. the subgame perfect Folk theorem holds with 1 – memory strategies.

While no further restrictions is needed for the third result to hold in the 2-player case, an additional restriction is needed for the first two: players must have common punishments. Moreover, we present robust examples in which there is a subgame perfect payoff that cannot be obtained by any 1 – memory subgame perfect strategy.

*This is a revised version of chapter 2 in Barlo (2003) and of Sabourian (1989). We thank Beth Allen, Pedro Amaral, Kemal Badur, Kevin Hasker, Ehud Kalai, Narayana Kocherlakota, Andy McLennan, Han Ozsoylev, Aldo Rusticini and Jan Werner and participants at the Applied Microeconomics Workshop and Mathematical Economics Workshop at the University of Minnesota, the Games 2002 conference, the Mid-West Mathematical Economics 2002 meeting, the SAET 2003 conference, and the ASSET 2003 conference for helpful comments and suggestions. Any remaining errors are ours.

1 Introduction

The Folk Theorem of repeated games states that any individually rational payoffs can be sustained as an equilibrium if the players are sufficiently patient (see Fudenberg and Maskin (1986) and Aumann and Shapley (1994)). Such a multiplicity of equilibria arises because in repeated games at any stage each player can condition their behavior on the past behavior of all the players. Such long memories are clearly unreasonable.

Even when players are impatient, equilibrium strategies often require the players to remember distant pasts. In fact, Abreu (1988) characterized the equilibrium outcome paths of discounted repeated games by using *simple* strategies which satisfy certain incentive conditions. Specifically, a simple strategy profile induces $n + 1$ outcome paths (states): the given prescribed play, and a punishment path for each of the n players. At any stage, unless there has been a single player deviation, simple strategies make the play continue along the given outcome path. In the case of a single player deviation, all the other players will punish the deviator with a player specific outcome path. Thus, in particular, the behavior at a given state of the game may depend unboundedly on the past. Therefore, because of the extensive memory dependence such simple equilibria are also often regarded as unappealing when compared with those in which the current behavior either does not depend on the past or depends at most on the behavior of the last few periods.

In this paper we ask whether or not we can obtain the equilibrium payoffs of repeated games by making use of strategies which depend only on what has happened in the previous period (*1 – memory strategies*) when the stage game has a "rich" number of actions. The answer is non-trivial as the examples we provide in section 3 identify an open neighborhood of normal form (stage) games and discount factors for which a subgame perfect payoff cannot be obtained with 1 – memory strategies.

Our main results demonstrate, at least for repeated games with more than 2 players, that strategies with 1-period memory are approximately enough to obtain all subgame perfect payoffs of discounted repeated games with complete information if the action space at each stage is sufficiently rich. More specifically, if the action spaces are connected then our approximation results hold, for games with more than two players, in the following three senses. First, any equilibrium payoff can be supported in contemporaneous ε – perfect equilibrium, for all $\varepsilon > 0$, with 1 – memory.[1] Second, if an equilibrium payoff profile is sustained by simple strategies with the property that the incentive conditions identified by Abreu (1988) hold *strictly*, then it can be approximated by 1 – memory equilibria. Third, for generic games, in the limit as the discount factor converges to one, any strictly individually rational payoff can be approximated by a 1 – memory equilibrium strategy profile. We also show that a similar result holds without the genericity assumption for the no discounting case. Furthermore, if 1-memory strategies can be conditioned on time, the result for the case of the discount factor converging is exact and not only an approximation.

The last sets of results with patient players also hold for 2–player games. However, with arbitrary discount factors, the first two sets of approximation results mentioned above (with ε – equilibria approximation or with simple strategy equilibria that satisfy the Abreu type incentive conditions strictly) do not necessarily extend to 2–player games without further assumptions on the structure of the equilibria. In particular, for arbitrary discount factors, we demonstrate that the same sets of results obtained for games with more than two players also hold for 2–player games if the equilibrium considered is such that the punishment path induced when one player deviates is the same as that induced when the other player deviates.

These results, then, suggest, at least for games with more than two player, that the restriction to 1 – memory strategies will not place severe limitations on equilibrium payoffs: It is approximately enough for players to remember what has happened in the previous period in order to obtain any subgame perfect payoff. Furthermore, our results also demonstrate that the Folk Theorem does not depend on the ability of players to remember more than the previous period. Thus, as long as players remember the last period imposing bounds on memory does not reduce the abundance of equilibrium payoffs.

Clearly, not all strategies are 1 – memory strategies. For any equilibrium outcome path (payoff profile), our approach involves employing simple strategies which first can be implemented by remembering only what has happened in the previous period, and second induce (approximately) the same equilibrium outome (payoffs). This requires each agent to identify the state of the play with 1 – memory. Such decoding of the state of play by observing the outcome in the previous period is

[1]The notion of contemporaneous perfect epsilon equilibrium, introduced and analyzed by Mailath, Postlewaite, and Samuelson (2005), demands that no single handed deviation pays strictly more than ε in any date and state. We often refer to this notion as ε–equilibrium.

feasible thanks to connected action spaces.

However, replacing memory with a complex set of actions is not sufficient to obtain all equilibria even with connected action spaces. Indeed, in section 3 we present an open neighborhood of 3-player stage games and discount factors such that any repeated game formed with these has a subgame perfect payoff vector that cannot be obtained by $1-$ memory equilibrium strategies. The action spaces for each player at any stage is convex and consists of the set of mixed strategies over two pure strategies. Moreover, we identify the following discontinuity around this particular payoff in the 3-player example: even though it cannot be sustained with $1-$ memory subgame perfect strategies, it can be supported by an $\varepsilon-$ equilibrium with $1-$ memory strategies for all $\varepsilon > 0$.

If the (simple) equilibrium strategy profile is such that the behavior at two states are different when the outcomes in the previous period are the same, then looking at the previous period is not sufficient to determine how the game should be played. In the case of a simple strategy profile, in order to rule out such confusing situations that prevent the equilibrium path and the punishment paths to be implementable with $1-$ memory, the action profiles used in the punishment phase for any player should occur neither on the equilibrium path nor be used in the punishment phase for other players, and any single deviation should be possible to be detected by observing the previous period. For instance, it must the case that a player being punished cannot, by deviating from the action that the punishments prescribe, give rise to an action profile on the equilibrium outcome.

When behavior depends only on the outcome in the preceding period, with two players there are additional confusing instances that might arise with a simple strategy profile that does not occur when the number of players exceed two: when n is three or above, it is considerably easier to identify single-player deviations than it is in the two player case. For instance, consider the following simple strategy in a 2–player game: the equilibrium path consists of repetitions of an action profile (a_1, a_2) and the punishment path for player 1 (resp., player 2) consists of repetitions of (b_1, b_2) (resp., (c_1, c_2)). When the punishments are not common and players observe (b_1, a_2) in the last period, they cannot conclude whether or not it was player 1 who deviated from the equilibrium path, or if it was player 2 who deviated from the punishment path of player 1.

Exploiting this observation in section 3 we present an open neighborhood of 2-player stage games and discount factors such that any repeated game constructed with these values possesses a subgame perfect payoff that cannot be obtained by any 1–memory equilibrium strategies. Moreover, the same conclusion holds for any payoff close to that particular one and sufficiently small $\varepsilon \geq 0$ in 1–memory ε-equilibrium.

In contrast, the above confusing instance cannot arise in a similar simple strategy in a three-player game. Indeed, if the equilibrium outcome consists of repetitions of (a_1, a_2, a_3) and punishment paths of players 1, 2 and 3 consist of repetitions of (b_1, b_2, b_3), (c_1, c_2, c_3) and (d_1, d_2, d_3), respectively, then players can use the action of player 3 in order to find out who has deviated: if they observe (b_1, a_2, a_3), then they conclude that player 1 has deviated, while if they observe (b_1, a_2, b_3), then it was player 2 that has deviated.

More formally, to rule out such ambiguities, we introduce a critical property, the notion of *confusion proofness* of simple strategies. This notion turns out to be both necessary and sufficient for players to find out in which phase of the $n + 1$ path the play is in (where n denotes the number of the players) by observing only what has happened in the previous period. In particular, our Proposition 1 establishes that a simple strategy is $1-$ memory if and only if it is *confusion-proof*.

We then establish our set of approximation results by showing that for any equilibrium payoff there exists an (ε –) equilibrium confusion proof simple strategy profile that approximates the original equilibrium in the three senses mentioned above. In particular, in the discounting case, payoffs of non-confusion proof simple strategies are approximated by making use of the notion of ε – *strict enforceability* of a simple strategy, $\varepsilon \geq 0$. This notion requires that at any date and state, every player loses less than $\varepsilon \geq 0$ by conforming with the simple strategy. Then, with the use of connected action spaces we employ this slack to construct a confusion proof simple strategy profile with a payoff arbitrarily close to the original one.[2] Indeed, this construction is the key ingredient for our discounting Folk theorem with $1-$ memory.

The above approach cannot be applied when agents have a small (finite) number of actions at each stage of the game, in which case $1-$ memory would not be enough for obtaining our results (see Sabourian (1998) which characterizes the set subgame perfect equilibria with bounded memory for the case of repeated games with no discounting and finite number of pure actions).[3] On the other

[2]The explanation for not being able to implement the equilibrium in the 3-player example with $1-$ memory and for the associated discontinuity is that there is no slack in the incentive conditions for the particular equilibrium payoff we consider; as a result there is no room to code information about the past into agents' behavior without violating the incentive conditions.

[3]Other important work in the literature on repeated games with limited memory include Kalai and Stanford (1988), Lehrer (1988), Aumann and Sorin (1989), Lehrer (1994), Neyman and Okada (1999), Bhaskar and Vega-Redondo (2002) and Barlo and Carmona (2006).

hand, rich (connected) action spaces endow the agents with the capacity to "code" information about who-deviated-when into their play; thereby, allow us to establish our results.

Notice, our richness of action space assumption is consistent with most standard games with infinite action spaces because it is often assumed that the action space is a convex (and hence connected) subset of some finite dimensional Euclidian space. Since the set of mixed strategies are also convex, it also follows that our richness assumption is also satisfied in any repeated game (with finite or infinite pure action space) in which at each stage the players are allowed to choose mixed strategies and past mixed actions are observable as in Aumann (1964).[4]

Several other papers study the effects of restricting the strategies players can use in repeated games: A particularly interesting case is that of 0 – memory strategies. In those, players play the same action profile in every period, independently of the past — this corresponds to the notion of *stationary strategies*. Clearly, in the class of games we consider, the stationary strategies that constitute subgame perfect equilibria are precisely those that consist of repeating a Nash action profile of the stage game forever, thus obviously, it cannot be hoped to characterize all Nash and subgame perfect payoffs using only 0 – recall strategies. Consequently, it is quite surprising that the next step of dependence on the past, 1–memory, is approximately able to characterize all equilibrium payoffs.

Another important class of repeated strategies are those represented by finite automata. Similar results to the ones obtained here appeared in Kalai and Stanford (1988), as they have shown that all subgame equilibrium payoffs can be approximately supported by finite automata as an approximate equilibrium for sufficiently large automata. They do not assume that the action space is large because they allow any finite size automata. Our results are different because we only consider strategies with *one* period recall.

Memory in terms of recall captures one aspect of complexity of a strategy. There are clearly other aspects of complexity of a strategy. We do not address these in this paper. In particular, we obtain our approximation results with 1-period memory/recall by using (cycle) paths that involve different action profiles at each date. Such paths may be complex if we use an alternative definition of complexity to period memory/recall. The objective here is not to tackle this general issue of complexity but simply to characterize the implications of recall restriction, and in particular, to

[4]We use the term "mixed action" to denote individual randomization over the actions in the stage game.

explain how, with some qualifications, in repeated games with rich action spaces players do not need to use much memory: remembering yesterday is almost enough to support all subgame perfect equilibrium payoffs.

In section 2, we provide the notation and the definitions. Section 3 presents two examples. Section 4 establishes when an outcome path can be obtained with the use of $1-$memory strategies. The discounting case is analyzed in Sections 5. In Section 6 we prove our $1-$memory Folk Theorems. Finally, Section 7 discusses time-dependent strategies.

2 Notation and Definitions

2.1 The stage game

A *normal form game* G is defined by $G = \left(N, (S_i)_{i\in N}, (u_i)_{i\in N}\right)$, where N is a finite set of players, S_i is the set of player i's actions and $u_i : \prod_{i\in N} S_i \to \mathbb{R}$ is player i's payoff function.

We assume that S_i is a connected and compact metric space and that u_i is continuous for all $i \in N$. Note that if S_i is convex, then S_i is connected. Therefore, the mixed extension of any finite normal form game satisfies the above assumptions.[5]

Let $S = \prod_{i\in N} S_i$ and $S_{-i} = \prod_{j\neq i} S_i$. Also, for any $i \in N$ denote respectively the *minmax payoff* and a *minmax profile* for player i by

$$v_i = \min_{s_{-i}\in S_{-i}} \max_{s_i\in S_i} u_i(s_i, s_{-i})$$

and

$$m_i \in \arg \min_{s_{-i}\in S_{-i}} \max_{s_i\in S_i} u_i(s_i, s_{-i}).$$

If G is a two-player game, a *mutual minmax profile* is $\bar{m} = (m_1^2, m_2^1)$.

2.2 Repeated game

The *supergame G^∞ of G* consists of an infinite sequence of repetitions of G.

We denote the action of any player i in G^∞ at any date $t = 1, 2, 3, \ldots$ by $s_i^t \in S_i$. Also, let $s^t = (s_1^t, .., s_n^t)$ be the profile of choices at t.

[5]More generally, the mixed extension of any normal form game with compact metric strategy spaces and continuous payoff functions also satisfies the above assumptions.

For $t \geq 1$, a t – *stage history* is a t – length sequence $h_t = (s^1, \ldots, s^t)$. The set of all t – stage histories is denoted by $H_t = S^t$ (the t – fold Cartesian product of S). We use H_0 to represent the initial (0 – stage) history. The set of all histories is defined by $H = \bigcup_{t=0}^{\infty} H_t$.

For all $i \in N$, player i's strategy is a function $f_i : H \to S_i$.[6] The set of player i's strategies is denoted by F_i, and $F = \prod_{i \in N} F_i$ is the joint strategy space with a typical element $f \in F$.

Given a strategy $f_i \in F_i$ and a history $h \in H$ we denote the *strategy induced at* h by $f_i|h$; thus $(f_i|h)(\bar{h}) = f_i(h, \bar{h})$, for every $\bar{h} \in H$. We will use $f|h$ to denote $(f_1|h, \ldots, f_n|h)$ for every $f \in F$ and $h \in H$.

Any strategy $f \in F$ induces an outcome at any date as follows:

$$\pi^1(f) = f(H_0) \text{ and } \pi^t(f) = f(\pi^1(f), \ldots, \pi^{t-1}(f)) \text{ for any } t > 1.$$

Denote the set of outcome paths by $\Pi = S \times S \times \cdots$ and define the outcome path induced by any strategy profile $f \in F$ by $\pi(f) = \{\pi^1(f), \pi^2(f), ...\} \in \Pi$.

We consider the following memory restriction on the set of strategies in this paper. For any history $h \in H$, the *1 – period tail* of h is $T(h) = s^t$ if $h = (s^1, \ldots, s^t)$ and $T(H_0) = H_0$.

Definition 1 *A strategy $f_i \in F_i$ has one period memory (henceforth called 1-memory) if $f_i(h) = f_i(\bar{h})$ for any two histories $h, \bar{h} \in H$ such that $T(h) = T(\bar{h})$.*

Notice that in the above definition the choice of action at any date t depends only on the last stage of the supergame and not t; thus 1-memory strategies are stationary and time-independent. We let F_i^1 be the set of all player i's strategies with 1-memory, and $F^1 = \prod_{i \in N} F_i^1$.

For all $i \in N$, let $U_i : F \to \mathbb{R}$ be player i's payoff function in the supergame of G.

Definition 2 *A strategy vector $f \in F$ is a* Nash equilibrium *of G^∞ if for all $i \in N$, $U_i(f) \geq U_i(\hat{f}_i, f_{-i})$ for all $\hat{f}_i \in F_i$. A strategy vector $f \in F$ is a* subgame perfect equilibrium *of G^∞ if $f|h$ is a Nash equilibrium for all $h \in H$.*

We also define a 1 – memory subgame perfect equilibrium as a subgame perfect equilibrium with the additional property that it has 1 – memory.[7]

[6]Notice that when S_i refers to the mixed extension of a normal form game, then the strategy in the repeated game at any period may depend on past randomization choices which in such cases must be publicly observable.

[7]Notice that with this definition the equilibrium strategy of each player has 1-memory but is best amongst *all* strategies, including those with memory longer than one. Alternatively, we could have just required optimality amongst the set of 1-memory strategies. For the purpose of the results in this paper the two possible definitions are equivalent.

3 Two Examples

In this section we present two examples of subgame perfect equilibrium payoffs that cannot be supported by a 1 – memory subgame perfect equilibrium strategy. The first example uses a two-player game while the second uses a three-player game.

3.1 Two Players

Consider the following finite normal form game:

$1\backslash 2$	a	b
a	4,4	2,5
b	5,2	0,0

We let $S_i = [0, 1]$ for all $i = 1, 2$, where $s_i \in S_i$ is to be interpreted as the probability assigned by player i to action a. Note that the minmax payoff is 2 for each player. In the case of player 1, it can only be obtained by $m^1 = (a, b)$, while $m^2 = (b, a)$ is the only action profile leading to the minmax payoff of player 2. Moreover, both m^1 and m^2 are Nash equilibria of the stage game. Finally, the mutual minmax profile is $\bar{m} = (m_1^2, m_2^1) = (b, b)$.

Now suppose that the above game is played infinitely often. If there are no bounds on the memory then the payoff of $(4, 4)$ is subgame perfect for any discount factor $\delta \geq 1/3$. To see this consider the following grim type pure strategy profile: (i) play (a, a) at each date on the equilibrium path, (ii) punish deviations from (a, a) by player $i = 1, 2$ by playing m^i forever (once in a punishment state playing m^i further deviations from m^i are ignored and the play continues with m^i). Clearly, at each date this strategy profile induces a payoff of 4 for each player. Furthermore, the profile constitutes a subgame perfect equilibrium: First, no player wants to deviate from the equilibrium path because

$$4 \geq (1 - \delta)5 + \delta 2 \text{ for any } \delta \geq 1/3, \tag{1}$$

and second, no player wants to deviate from m^i, $i = 1, 2$, since he is playing a best reply.

Now the above subgame perfect strategy profile cannot be implemented with 1-period memory, even though the minmax action profiles for both agents are Nash equilibria of the stage game. This is because the punishment of minmaxing a deviator creates confusing instances if at each stage the players can only recall the outcome of the previous period. For example, the strategy profile is ill-defined with 1-memory if $m^1 = (b, a)$ is observed: it cannot be inferred if in the previous period player 1 has deviated from (a, a) or if player 2 was being punished.

The impact of the 1 – memory restriction is in fact more profound than not being able to implement the above strategy profile. In fact, we can show that there does not exist any subgame perfect strategy profile with 1 – memory that induces an average payoff of $(4, 4)$ for $\delta = 1/3$. This holds even when mixed strategies are observable. In appendix A.1 we also prove that this conclusion is robust to perturbations in the discount factor, and payoffs simultaneously.

To establish this result suppose otherwise; then there exists a 1 – memory subgame perfect equilibrium f that induces an average payoff of $(4, 4)$ for $\delta = 1/3$. Now since we assume that mixed strategies are observable and f has 1 – memory, it follows that there exist functions g_i : $[0, 1]^2 \cup \{H_0\} \to [0, 1]$ for all $i = 1, 2$ such that $f_i(h) = g_i(T(h))$.

Now we obtain a contradiction in several steps. First, we show that the only way to obtain an average payoff of $(4, 4)$ is for f to play (a, a) repeatedly forever, i.e. it must be that $g_i(h_0) = g_i(1, 1) = 1$ for all $i = 1, 2$. Let $p_1 = g_1(h_0)$, $q_1 = g_2(h_0)$, $p_t = g_1(p_{t-1}, q_{t-1})$ and $q_t = g_2(p_{t-1}, q_{t-1})$ for all $t \in \mathbb{N}$. Since $U_1(f) = U_2(f) = 4$, it follows that

$$8 = U_1(f) + U_2(f) = \frac{2}{3} \sum_{t=1}^{\infty} \frac{8p_t q_t + 7(p_t(1 - q_t) + q_t(1 - p_t))}{3^{t-1}}. \tag{2}$$

Now $8p_t q_t + 7(p_t(1 - q_t) + q_t(1 - p_t)) \leq 8$. Therefore, condition (2) holds only if $p_t = q_t = 1$ for all $t \in \mathbb{N}$. But this implies that $g_i(h_0) = g_i(1, 1) = 1$ for all $i = 1, 2$.

Next, we show that if player 2 were to deviate from (a, a) by playing b, player 1 must punish by assigning a zero probability to a in the period following the deviation: $g_1(1, 0)$ must equal 0. This is because, since player 2 can guarantee himself a payoff of 2 in every period, this deviation would at least yield him a return of $(1 - \delta)5 + (1 - \delta)\delta(4g_1(1, 0) + 2(1 - g_1(1, 0))) + 2\delta^2 = 4 + 4/9g_1(1, 0)$; thus, this deviation is not profitable only if $g_1(1, 0) = 0$. By a symmetric argument, $g_2(0, 1) = 0$.

When the play in period 1 is $(1, 0)$, we know that in the next period player 1 must play b. However, this is rational only if $g_2(1, 0)$ is high, otherwise player 1 would be tempted to play a instead of playing b. In fact, we show next that $g_2(1, 0)$ must be at least 1/6 in order for player 1 to punish player 2. To see this, consider for player 1 the strategy of playing a in every history: $\bar{f}_1(h) = 1$ for all $h \in H$. Then,

$$U_1(\bar{f}_1, f_2|(1, 0)) \geq (1 - \delta)(2g_2(1, 0) + 2) + 2\delta = 2 + \frac{4g_2(1, 0)}{3}.$$

Also, we have that

$$\begin{aligned} U_1(f|(1,0)) &\leq (1-\delta)u_1(g_1(1,0), g_2(1,0)) + 5\delta \\ &= (1-\delta)5g_2(1,0) + 5\delta = \frac{10g_2(1,0)+5}{3}. \end{aligned}$$

Since f is a subgame perfect equilibrium, $U_1(f|(1,0)) \geq U_1(\bar{f}_1, f_2|(1,0))$. Hence, $g_2(1,0) \geq 1/6$.

At this point the difference between the full memory and 1–memory case is clear: In the full memory case a single deviation by player 1 from playing a leads player 2 to choose b forever, while in the 1–memory case although it leads player 2 to b in the first period after the deviation, in the second period after the deviation player 2 would have to play a with a probability of at least 1/6 if 1 plays played a in the first period after the deviation.

Consequently, the punishment with 1–memory is less severe. This implies that a profitable deviation for player 1 exists: First player 1 chooses b, and then a forever. We obtain the required contradiction since this deviation delivers player 1 a return of at least

$$5(1-\delta) + 2\delta(1-\delta) + (4g_2(1,0) + 2(1-g_2(1,0)))\delta^2(1-\delta) + 2\delta^3 = 4 + \frac{4g_2(1,0)}{27} \geq 4 + \frac{2}{81} > 4.$$

3.2 Three Players

Let G be the mixed extension of the following normal form game with three players: all players have pure action spaces given by $A_i = \{a, b\}$,

$$u_3(a_1, a_2, a_3) = \begin{cases} 4 & \text{if } a_3 = a \text{ and} \\ 2 & \text{if } a_3 = b. \end{cases}$$

for all $a_1 \in A_1$ and $a_2 \in A_2$, u_1 and u_2 are defined by Table 1 above if $a_3 = a$ and arbitrarily if $a_3 = b$.

Arguing as in the previous section, one can show that $(4, 4, 4)$ is a subgame perfect equilibrium payoff that cannot be supported by a $1-$memory subgame perfect equilibrium for $\delta = 1/3$. Indeed, if f is a $1-$memory subgame perfect equilibrium, then $f_3(h) = a$ for all $h \in H$ and so we are effectively in the same situation as in the above subsection.

4 Confusion-Proof Paths and 1-Memory

Following Abreu (1988), $f \in F$ is *a simple strategy profile represented by* $n+1$ *paths* $(\pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(n)})$ if f specifies: (i) play $\pi^{(0)}$ until some player deviates singly from $\pi^{(0)}$; (ii) for any $j \in N$, play $\pi^{(j)}$ if the jth player deviates singly from $\pi^{(i)}$, $i = 0, 1, \ldots, n$, where $\pi^{(i)}$ is the ongoing previously specified path; continue with the ongoing specified path $\pi^{(i)}$, $i = 0, 1, \ldots, n$, if no deviations occur or if two or more players deviate simultaneously. These strategies are simple because the play of the game is always in only $(n+1)$ *states*, namely, in state $j \in \{0, \ldots, n\}$ where $\pi^{(j),t}$ is played, for some $t \in \mathbb{N}$. In this case, we say that the play is in *phase t of state j*. A profile $(\pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(n)})$ of $n+1$ outcome paths is *subgame perfect* if the simple strategy represented by it is a subgame perfect equilibrium .

Henceforth, when the meaning is clear, we shall use the term $(\pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(n)})$ to refer to both an $n+1$ outcome paths as well as to the simple strategy profile represented by the paths. Also, when referring to a profile of $n+1$ outcome paths, we shall not always explicitly mention $n+1$ and simply refer to it by a profile of outcome paths.

Abreu (1988) used the concept of simple strategies to characterize the set of subgame perfect equilibria. In this section, we consider simple strategy profiles that can be implemented with $1-$ memory. For this purpose, we introduce the notion of a confusion-proof profile of outcome paths and show in Proposition 1 below that that a profile of outcome paths can be supported by a $1-$ memory simple strategy if and only if it is confusion proof. The construction used in Proposition 1 is our main tool and is used throughout the paper.

The notion of a confusion-proof profile of outcome paths is motivated by the following observations. For a profile of simple strategies to be supported by a $1-$ memory simple strategy the players need to find out the correct state of the play by only observing the action profile in the previous period. This clearly is not always possible. To see this consider a simple strategy represented by the profile of paths $(\pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(n)})$. Then, three kinds of complications can arise if the strategies have $1-$ memory.

The first kind of complication happens when

$$\pi^{(i),t} = \pi^{(j),r} \text{ for some } i, j \in \{0, 1, \ldots, n\} \text{ and } t, r \in \mathbb{N}. \tag{3}$$

That is, the action profile in phase t of state i is the same as that in phase r of state j . Since players condition their behavior only last period's action profile, the players cannot distinguish between phase t of state i and phase r of state j, and therefore the simple strategy cannot be implemented, unless $\pi^{(i),t+1} = \pi^{(j),r+1}$.

The second kind of complication arises when

$$\pi_{-k}^{(i),t} = \pi_{-k}^{(j),r} \text{ for some } i, j \in \{0, 1, \ldots, n\}, k \in N \text{ and } t, r \in \mathbb{N}. \tag{4}$$

In words, every player other than $k \in N$ takes the same action in phase t of state i and in phase r of state j. Then if, for example, the last period's action profile is $\pi^{(j),r}$, the players would not be able to deduce whether the play in the previous period was in phase t of state i and player k deviated to $\pi_k^{(j),r}$ or whether it was in phase r of state j and no deviation occur. Since a deviation by player k from $\pi_k^{(i),t}$ to $\pi_k^{(j),r}$ in phase t of state i is impossible to be detected by observing only the action in the last period, the simple strategy cannot be implemented, unless $\pi^{(i),t+1} = \pi^{(j),r+1} = \pi^{(k),1}$.

The third kind of complication appears when

$$\pi_{-l,m}^{(i),t} = \pi_{-l,m}^{(j),r} \text{ for some } i, j \in \{0, 1, \ldots, n\}, l, m \in N \text{ and } t, r \in \mathbb{N}. \tag{5}$$

In words, all players other than l and $m \in N$ take the same action both in phase t of state i, and in phase r of state j. Then, if the last period's action profile is given by $(\pi_l^{(i),t}, \pi_m^{(j),r}, (\pi_k^{(i),t})_{k \neq l,m}) = (\pi_l^{(i),t}, \pi_m^{(j),r}, (\pi_k^{(j),r})_{k \neq l,m})$, players, looking back one period, can conclude that either player l or player m has deviated. But, they cannot be certain of the identity of the deviator. Consequently, both of them must be punished. This requires $\pi^{(l)} = \pi^{(m)}$.

These observations are formalized below as follows. For any profile of outcome paths $(\pi^{(0)}, \ldots, \pi^{(n)}) \subseteq \Pi^{n+1}$, let

$$\Omega\left(\{i, t\}, \{j, r\}\right) = \{k \in N : \pi_k^{(i),t} \neq \pi_k^{(j),r}\},$$

be the set of players whose actions in phase t of stage i and in phase r stage j are different.

Definition 3 *A profile* $(\pi^{(0)}, \ldots, \pi^{(n)}) \in \Pi^{n+1}$ *of outcome paths is* confusion-proof *if for any* $i, j \in \{0, 1, \ldots, n\}$ *and* $t, r \in \mathbb{N}$ *the following holds:*

1. *if* $\Omega(\{i, t\}, \{j, r\}) = \emptyset$*, then* $\pi^{(i),t+1} = \pi^{(j),r+1}$*.*
2. *if* $\Omega(\{i, t\}, \{j, r\}) = \{k\}$ *for some* $k \in N$*, then* $\pi^{(i),t+1} = \pi^{(j),r+1} = \pi^{(k),1}$*.*
3. *if* $\Omega(\{i, t\}, \{j, r\}) = \{k, l\}$ *for some* k *and* $l \in N$*, then* $\pi^{(k)} = \pi^{(l)}$*.*

The above observations, which motivated the definition of confusion-proof outcome paths, suggest that confusion-proofness is necessary to support a profile of outcome paths with an $1-$memory simple strategy. The following Theorem asserts that confusion-proofness is, in fact, not only a necessary but also a sufficient condition to support a profile of outcome paths with an $1-$memory simple strategy.

Proposition 1 *A profile of outcome paths is confusion-proof if and only if there exists a 1 – memory simple strategy represented by it.*

The 1 – memory strategy f supporting the confusion-proof profile of outcome paths $(\pi^{(0)}, \ldots, \pi^{(n)})$ is as follows: If the last period of a given history equals $\pi^{(j),t}$, for some $j = 0, 1, \ldots, n$ and $t \in \mathbb{N}$, then player i chooses $\pi_i^{(j),t+1}$. If only player $k \in N$ deviated from the outcome $\pi^{(j)}$ in the last period of the history, then player i chooses $\pi_i^{(k),1}$. Finally, if more then one player deviated from the outcome $\pi^{(j)}$ in the last period of the history, then player i chooses $\pi_i^{(j),t+1}$, i.e., such deviations are ignored. Since f has 1 – memory and has the structure of a simple strategy, we say that f is a *1 – memory simple strategy*. As before, the profile $(\pi^{(0)}, \ldots, \pi^{(n)})$ represents f. The main task of the sufficiency part of the proof of Proposition 1 is to show that f is well defined; this is the case since $(\pi^{(0)}, \ldots, \pi^{(n)})$ is confusion-proof.

Before turning to the equilibrium characterization with 1-memory, we shall next provide a set of easily tractable sufficient conditions for a profile of outcome paths to be confusion-proof.

Lemma 1 *A profile of outcome paths $(\pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(n)})$ is confusion-proof if one of the following conditions hold:*

1. *$n \geq 3$, and for all $i, j \in \{0, 1, \ldots, n\}$ and $t, r \in \mathbb{N}$ satisfying $(i, t) \neq (j, r)$ there exist at least three players $l \in N$ such that*
$$\pi_l^{(i),t} \neq \pi_l^{(j),r}; \tag{6}$$

2. *$n = 2$ and*

   (a) players have the same punishment path, i.e.,
$$\pi^{(1)} = \pi^{(2)}; \tag{7}$$

   (b) for all $i, j \in \{0, 1\}$ and $t, r \in \mathbb{N}$ satisfying $(i, t) \neq (j, r)$ and for any $l = 1, 2$
$$\pi_l^{(i),t} \neq \pi_l^{(j),r}. \tag{8}$$

The condition given in the above lemma for the case of three or more players is clearly sufficient for confusion-proofness. Indeed, if $(\pi^{(0)}, \ldots, \pi^{(n)})$ satisfies (6), then for all $i, j \in \{0, \ldots, n\}$ and $t, r \in \mathbb{N}$ such that $(i, t) \neq (j, r)$, it follows that $|\Omega(\{i, t\}, \{j, r\})| \geq 3$. Thus, $(\pi^{(0)}, \ldots, \pi^{(n)})$ is confusion-proof in a trivial way. Similarly, if $(\pi^{(0)}, \pi^{(1)}, \pi^{(2)})$ satisfies (7) and (8) in a game with two players, then for all $i, j \in \{0, 1, 2\}$ and $t, r \in \mathbb{N}$ such that $(i, t) \neq (j, r)$, it follows that $|\Omega(\{i, t\}, \{j, r\})| = 2$, except when $i, j \in \{1, 2\}$. This together with (7) imply that $(\pi^{(0)}, \pi^{(1)}, \pi^{(2)})$ is confusion-proof when $n = 2$.

Conditions (6) and (8), however, are not necessary for confusion-proofness in the case when $n > 2$ and $n = 2$, respectively. For instance, $(\pi^{(0)}, \dots, \pi^{(n)})$ defined by $\pi^{(j),t} = s \in S$ for all $j \in \{0, \dots, n\}$ and $t \in \mathbb{N}$ is confusion proof but it does not satisfy those conditions. It is also the case that condition (7) is not necessary for confusion-proofness. For example, let $(\pi^{(0)}, \pi^{(1)}, \pi^{(2)})$ be defined by $\pi^{(j),t} = s \in S$ for all $j \in \{0, 1\}$ and $t \in \mathbb{N}$ and by $\pi^{(2),t} = \begin{cases} (\bar{s}_1, s_2) & \text{if } t = 1, \\ s & \text{if } t \geq 2. \end{cases}$. Then, $(\pi^{(0)}, \pi^{(1)}, \pi^{(2)})$ is confusion-proof but $\pi^{(1)} \neq \pi^{(2)}$. However, as the following remark demonstrates, with $n = 2$ the above example is the only possible confusion proof paths that violates condition (7) and therefore identical punishment paths for both players is almost necessary for confusion proofness in 2-player games.

Remark 1 *If $n = 2$ and $(\pi^{(0)}, \pi^{(1)}, \pi^{(2)})$ is confusion-proof, then either $\pi^{(1)} = \pi^{(2)}$ or there exists $i \in N$, $s_i, \bar{s}_i \in S_i$ and $s_{-i} \in S_{-i}$ such that $\pi^{(i),t} = (s_i, s_{-i})$ for all $t \in \mathbb{N}$ and*

$$\pi^{(-i),t} = \begin{cases} (\bar{s}_i, s_{-i}) & \text{if } t = 1, \\ (s_i, s_{-i}) & \text{if } t \geq 2. \end{cases}$$

As it clear from the above, the analysis of the confusion proof simple paths, and hence 1-memory strategies, is considerably different for the case of 2-player games from that with three or more players. The basic difference between two cases is similar to that found in the implementation literature. Here, as in there, when there are only two players, it may not be possible to detect which of the two players have deviated and as a result both must be punished with the same punishement path whenever a deviation is detected.

5 Discounting

In this section, we assume that all agents discount the future returns by a common discount factor $\delta \in (0, 1)$. Thus the payoff in the supergame $G^\infty(\delta)$ of G is now given by

$$U_i(f) = (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_i(\pi^t(f)).$$

Also, for any $\pi \in \Pi$, $t \in \mathbb{N}$, and $i \in N$, let $V_i^t(\pi) = (1 - \delta) \sum_{r=t}^{\infty} \delta^{r-t} u^i(\pi^r)$ be the continuation payoff of player i at date t if the outcome path π is played. For simplicity, we write $V_i(\pi)$ instead of $V_i^1(\pi)$.

An outcome path π is a *subgame perfect outcome path* if there exists a subgame perfect equilibrium f such that $\pi = \pi(f)$.

A profile of outcome paths $(\pi^{(0)}, \dots, \pi^{(n)}) \in \Pi^{n+1}$ is *weakly enforceable* if

$$V_i^t(\pi^{(j)}) \geq (1-\delta) \max_{s_i} u_i(s_i, \pi_{-i}^{(j),t}) + \delta V_i(\pi^{(i)}) \tag{9}$$

for all $i \in N$, $j \in \{0, 1, .., n\}$ and $t \in \mathbb{N}$.

¿From Abreu (1988), it is well known that weak enforceability is equivalent to subgame perfection. More precisely, an outcome path $\pi^{(0)}$ is a subgame perfect outcome path if and only if there exists a simple strategy represented by a weakly enforceable profile of outcome paths $(\pi^{(0)}, \dots, \pi^{(n)})$.

In our setting we note that, by Proposition 1, any weakly enforceable, confusion-proof profile of outcome paths can be supported by a $1-$ memory simple subgame perfect equilibrium strategy. In particular, the same holds for any subgame perfect payoff vector that can be obtained by a confusion-proof profile of outcome paths. Moreover, it is worthwhile to note that connectedness of strategy spaces is not needed for these conclusions that are summarized in the following corollary to Proposition 1.

Corollary 1 *Let u be subgame perfect equilibrium payoff vector that can be supported by a weakly enforceable, confusion-proof profile of outcome paths. Then, there is a $1-$ memory subgame perfect equilibrium strategy f such that $U(f) = u$.*

In general, as was shown by the examples in Section 3, we cannot exactly support all subgame perfect payoff vectors by $1-$ memory subgame perfect equilibrium strategies. In fact, the best that can be hoped for is to obtain them approximately. There are three aspects involved in our approximations. The first involves the equilibrium concept in question. To that regard, we employ the following notion of equilibria, introduced in Mailath, Postlewaite, and Samuelson (2005):

For all $\varepsilon \geq 0$, a strategy profile $f \in F$ is a *contemporaneous $\varepsilon-$ Nash equilibrium* of the supergame of G if for all $i \in N$, $V_i^t(\pi(f)) \geq V_i^t(\pi(\hat{f}_i, f_{-i})) - \varepsilon$ for all $t \in \mathbb{N}$ and $\hat{f}_i \in F_i$. A strategy vector $f \in F$ is a *contemporaneous $\varepsilon-$ subgame perfect equilibrium* of the supergame of G if $f|h$ is a *contemporaneous* $\varepsilon-$ Nash equilibrium for every $h \in H$.[8]

[8]We should note that any contemporaneous $\varepsilon-$ Nash equilibrium of the supergame of G is an $\varepsilon-$ Nash equilibrium of the supergame of G, and the reverse is not true. Similarly, any contemporaneous $\varepsilon-$ subgame perfect equilibrium of the supergame of G is an $\varepsilon-$ subgame perfect equilibrium of the supergame of G, and the reverse is not true.

The second kind of approximation features the distance in the payoff space, i.e. given a subgame perfect payoff vector, can a payoff vector close to it be sustained as an equilibrium with 1-memory?

The third and widely used approximation concerns the discount factor: any strictly individual rational payoff can be supported in subgame perfect equilibrium for δ sufficiently large? Does the same hold with 1-memory restriction?

Since any subgame perfect equilibrium has to be weakly enforceable, it turns out that a slack in the incentive equations (9) is needed in order to perform the required approximations. This leads us to introduce the notion ε – strictly enforceability of a subgame perfect equilibrium payoff vector.

Definition 4 *For all $\varepsilon \geq 0$, a payoff vector u is a ε – strictly enforceable subgame perfect equilibrium payoff if there exists a $n+1$ outcome paths $(\hat{\pi}^{(0)}, \ldots, \hat{\pi}^{(n)})$ such that $V_i(\hat{\pi}^{(0)}) = u_i$ and*

$$\inf_{t \in \mathbb{N}} \left(V_i^t(\hat{\pi}^{(j)}) - \left((1-\delta) \max_{s_i} u_i(s_i, \hat{\pi}_{-i}^{(j),t}) + \delta V_i(\hat{\pi}^{(i)}) \right) \right) > -\varepsilon \tag{10}$$

for all $i \in N$ and all $j \in \{0, 1, \ldots, n\}$.

Thus, payoff vector is an ε – strictly enforceable subgame perfect equilibrium payoff if it can be obtained with a contemporaneous ε – subgame perfect equilibrium simple strategy profile. Therefore, in such an equilibrium at any date, any player loses less than $\varepsilon \geq 0$ if he continues at the equilibrium path, and also if he continues to punish any player with the player specific punishment paths given by the simple strategy in question.

For convenience, if u is an ε – strictly enforceable subgame perfect equilibrium payoff with $\varepsilon = 0$, we simply say that u is a strictly enforceable subgame perfect equilibrium payoff.

Theorem 1 demonstrates that under certain conditions, any ε-strictly enforceable subgame perfect equilibrium payoff can be approximately supported with a 1 – memory contemporaneous ε – subgame perfect equilibrium.

Theorem 1 *Let $\varepsilon \geq 0$ and u be an ε – strictly enforceable subgame perfect equilibrium payoff induced by the simple strategy profile $(\hat{\pi}^{(0)}, \ldots, \hat{\pi}^{(n)})$. Then, for every $\eta > 0$ there is a 1 – memory contemporaneous ε – subgame perfect equilibrium strategy f such that $|U(f) - u| < \eta$, provided that either $n \geq 3$, or $n = 2$ and $\hat{\pi}^{(1)} = \hat{\pi}^{(2)}$.*

Whenever $\varepsilon = 0$, Theorem 1 shows that every neighborhood of any strictly enforceable subgame perfect payoff profile contains a payoff profile that can be obtained with a 1 – memory subgame perfect if either $n \geq 3$, or $n = 2$ and $\hat{\pi}^{(1)} = \hat{\pi}^{(2)}$. Thus, in these cases, we can approximate a payoff profile using subgame perfection with 1-memory, if players strictly prefer to follow the associated simple strategy at every subgame.

Although there are subgame perfect equilibrium payoff vectors which are not strictly enforceable (not satisfying Definition 4 for $\varepsilon = 0$), note that any subgame perfect equilibrium payoff profile is a contemporaneous $\varepsilon -$ perfect equilibrium payoff vector, for all $\varepsilon > 0$. This simple observation implies the following corollary to Theorem 1.

Corollary 2 *For every subgame perfect equilibrium payoff u, and every $\eta > 0$, there is a $1 -$ memory contemporaneous $\eta-$ perfect equilibrium strategy f such that $|U(f) - u| < \eta$, whenever either $n \geq 3$, or $n = 2$ and there exist a subgame perfect profile of simple outcome paths $(\hat{\pi}^{(0)}, \hat{\pi}^{(1)}, \hat{\pi}^{(2)})$ such that $\hat{\pi}^{(1)} = \hat{\pi}^{(2)}$ and $V_i(\hat{\pi}^{(0)}) = u_i$ for all $i \in N$.*

This corollary to Theorem 1 is in the same spirit as Theorem 4.1 of Kalai and Stanford (1988). It shows that with three more players, given any $\eta > 0$, every subgame perfect equilibrium payoff vector can be approximately obtained by a $1 -$ memory contemporaneous $\eta -$ subgame perfect equilibrium. In other words, the value of any recall beyond observing the last period, is arbitrarily small. Moreover, the same conclusion holds for two-player games if the punishement paths needed to enforce the original equilibrium are the same for both players.

The proof of Theorem 1 involves showing that under the assumptions of the Theorem, together with S_i being connected and u_i being continuous, the given profile of outcome paths $(\hat{\pi}^{(0)}, \ldots, \hat{\pi}^{(n)})$ supporting u as an $\varepsilon -$ strictly enforceable subgame perfect equilibrium payoff vector can be approximated by a confusion-proof profile of outcome paths $(\bar{\pi}^{(0)}, \ldots, \bar{\pi}^{(n)})$, that is arbitrarily close to the first in terms of the distance in payoffs. This implies that the latter profile of outcome paths is also an $\varepsilon -$ strictly enforceable subgame perfect equilibrium. Applying Proposition 1 completes the proof.

The relation of the examples presented in section 3 with these results reveals very interesting findings. Recall that we identified a subgame perfect (and Pareto optimal) payoff in the 3–player case which cannot be supported by any $1 -$ memory equilibrium strategy. On the other hand, Corollary 2 displays that the same payoff can arbitrarily closely be approximated in contemporaneous $\varepsilon-$ equilibrium with $1 -$ memory for all $\varepsilon > 0$. Indeed, it is not very difficult to show that a stronger result holds, and this payoff can be exactly obtained in the same equilibrium concept for all $\varepsilon > 0$. Therefore, these imply that there is a *discontinuity* in the following sense: Even though that subgame perfect payoff can be obtained with 1 – memory in contemporaneous ε–equilibrium for all $\varepsilon > 0$, it cannot be exactly sustained with subgame perfection and 1 – memory. Moreover, it should be pointed out that the hypothesis of Theorem 1 does not apply because this payoff cannot be obtained with any strictly enforceable simple strategy.

Considering the 2–player example of section 3 it should be noticed that the hypothesis of Corollary 2 (and hence, those of Theorem 1) are not satisfied: The punishments are not common. Indeed, in appendix A.2 we first prove that there is an open neighborhood of a subgame perfect (and Pareto optimal) payoff such that no subgame perfect payoff in there can be obtained by 1 – memory subgame perfect strategies. Moreover, we also prove that a similar result holds when ε – equilibrium is used.

As we mentioned before, Theorem 1 establishes that any strictly enforceable utility vector can be *approximated* by a 1 – memory subgame perfect payoff vector. Can such 1 – memory implementation of strictly enforceable utility be exact? Clearly, if a strictly enforceable utility vector u has the additional property that it can be obtained by a confusion proof profile of outcome paths, $(\hat{\pi}^{(0)}, \ldots, \hat{\pi}^{(n)})$, then there exists a 1 – memory simple strategy profile that supports u exactly. Even if $(\hat{\pi}^{(0)}, \ldots, \hat{\pi}^{(n)})$ is not immune to confusion, but is strictly enforceable and the single path $\hat{\pi}^{(0)}$ is implementable with 1 – memory ($\hat{\pi}^{(0)}$ does not involve any confusing instances), u can still be sustained exactly by a 1 – memory equilibrium profile. As in the proof of Theorem 1, this can be established by constructing another punishment profile $(\bar{\pi}^{(1)}, \ldots, \bar{\pi}^{(n)})$ such that $(\hat{\pi}^{(0)}, \bar{\pi}^{(1)}, \ldots, \bar{\pi}^{(n)})$ is confusion-proof simple equilibrium and hence 1 – memory implementable.[9] To show this formally, we shall next define a confusion-proof single outcome path and a confusion-proof payoff vector.

A single outcome path π is free of confusion if it satisfies the following three conditions. First, if the vector of actions is the same in two different periods, then the action profile in the period following one of those periods must equal the action profile following the other period. Second, it cannot be the case that there is exactly one player whose action is different in two given periods. And third, with more than two players, it cannot be the case that there is exactly two players whose actions are different in two given periods. These restrictions are described in Definition 5 below, and are similar to those made in Definition 3. As before, the set of players that in a given outcome path play a different actions in different periods is a key concept. Formally, for any outcome paths

[9]In the next section, we shall use this observation to establish our Folk Theorem type results with 1 – memory.

$\pi \in \Pi$ this set is defined by

$$\Omega(t,r) = \{i \in N : \pi_i^t \neq \pi_i^r\} \text{ for any two periods } t \text{ and } r \in \mathbb{N}.$$

Definition 5 *A single path $\pi \in \Pi$ is confusion-proof if the following properties hold:*

1. *$|\Omega(t,r)| = 0$ for some $t, r \in \mathbb{N}$, implies $\pi^{t+1} = \pi^{r+1}$.*
2. *There are no $t, r \in \mathbb{N}$ with $|\Omega(t,r)| = 1$.*
3. *If $n > 2$, then $|\Omega(t,r)| \neq 2$ for all $t, r \in \mathbb{N}$.*

We define the set of Π^{cp} to be the set of all single confusion proof paths. Furthermore, a payoff vector u is confusion-proof if it can supported by a single confusion-proof path: there exists $\pi \in \Pi^{cp}$ such that $V(\pi) = u$.

Using the same techniques as in the proof of Theorem 1 one can prove the following Proposition.

Proposition 2 *Suppose that u is supported by a single confusion proof path $\pi^{(0)} \in \Pi^{cp}$. Assume also that there exists n paths $(\pi^{(1)}, \ldots, \pi^{(n)}) \in \Pi^n$ such that $(\pi^{(0)}, \pi^{(1)}, \ldots, \pi^{(n)})$ is strictly enforceable. Then, there is a $1-$memory subgame perfect equilibrium strategy f such that $U(f) = u$, provided that either $n \geq 3$, or $n = 2$ and $\pi^{(1)} = \pi^{(2)}$.*

Thus, confusion-proof strictly enforceable subgame perfect equilibrium payoffs are most well-suited for our goal of supporting payoff vectors by $1-$memory subgame perfect equilibrium strategies. This makes it natural to ask whether we can describe the set of such payoffs. The following section will provide a partial description of that set, for all sufficiently large discount factors. This implies that all payoff vector in the class we identify can be supported by a $1-$memory subgame perfect equilibrium strategies provided that the discount factor is sufficiently large.

6 Folk Theorems

6.1 Individually Rational Payoffs

Let $D = \text{co}\,(u(S))$, $\mathcal{U} = \{y \in D : y_i \geq v_i \text{ for all } i \in N\}$ and $\mathcal{U}^0 = \{y \in D : y_i > v_i \text{ for all } i \in N\}$. The set $\mathcal{U}$ (resp. $\mathcal{U}^0$) is *the set of (resp. strictly) individually rational payoffs.* Clearly, $D = \text{co}(u(S))$ is compact, since $u(S)$ is compact.

Fudenberg and Maskin (1986) establish the Folk Theorem for repeated games by assuming that $\mathcal{U}$ has full-dimension ($\dim(\mathcal{U}) = n$). Next, we note some implications of full-dimensionality.

Lemma 2 *If* $\dim(\mathcal{U}) = n$*, then:*

1. *there exists* $\bar{u} \in \text{int}(\mathcal{U}^0)$*,*
2. $\mathcal{U} = \overline{\text{int}(\mathcal{U}^0)}$*, and*
3. *for all* $\alpha \in \mathcal{U}^0$ *and* $i \in N$*, there exists* $y^i \in \mathcal{U}^0$ *such that* $y^i_i < \alpha_i$ *and* $y^i_i < y^j_i$ *for all* $i, j \in N$ *with* $j \neq i$*.*

While the first two properties follow easily from Theorems 6.1 and 6.2 in Rockafellar (1970), the third has been established in Abreu, Dutta, and Smith (1994).

6.2 Folk theorem with discounting

For all $\alpha \in \mathcal{U}^0$ let $\Lambda(\alpha, \delta) \subseteq \Pi$ be the set of all single confusion-proof outcome paths $\pi \in \Pi^{cp}$ such that

$$V^t_i(\pi, \delta) \geq \alpha_i \text{ for all } i \in N \text{ and } t \in \mathbb{N}.$$

Also, let denote the set of confusion-proof payoff vectors that is supported by the set $\Lambda(\alpha, \delta)$ by

$$C(\alpha, \delta) = \{u \in \mathbb{R}^n : u = V(\pi, \delta) \text{ for some } \pi \in \Lambda(\alpha, \delta)\}.$$

The next proposition shows that all payoffs in $C(\alpha, \eta)$ can be supported by 1 – memory subgame perfect equilibria if players are sufficiently patient.

Proposition 3 *Suppose that either* $\dim(\mathcal{U}) = n$ *or* $n = 2$ *and* $\mathcal{U}^0 \neq \emptyset$*. Then for all* $\alpha \in \mathcal{U}^0$*, there exists* $\bar{\delta} \in (0, 1)$ *such that for all* $\delta \geq \bar{\delta}$ *the following holds: for all payoffs* $u \in C(\alpha, \delta)$ *there exists 1 – memory subgame perfect equilibrium strategy* f *with* $U(f, \delta) = u$*.*

The proof of Proposition 3 is similar that of Fudenberg and Maskin (1986) except that we do not allow for correlated strategies. We note that with correlated strategies, Proposition 3 would be completely standard since any payoff vector in D is confusion-proof (can be obtained by the repetition of the same actions) and so the conclusion would follow from Theorem 2 in Fudenberg and Maskin (1986). Hence, Proposition 3 is only interesting provided that players cannot use correlated strategies.

Despite this, our method is similar to theirs. For the case of $n = 3$ the payoff u is supported by a simple strategy with equilibrium path given by π as above. Then, the punishment path for each player $i \in N$ is as follows: play m^i for $T - 1$ periods and then play a path $\bar{\pi}^{(i)}$. The path

$\bar{\pi}^{(i)}$ is chosen so that its payoff $V_i(\bar{\pi}^{(i)})$ for player i to have the following four properties. First, it is strictly below the payoff i receives on the equilibrium path at any date (i.e., $V_i(\bar{\pi}^{(i)}) < \inf_t V_i^t(\pi)$). Second, it strictly exceeds the minmax payoff for i. Third, it is below its continuation payoff at any date (i.e., $V_i(\bar{\pi}^{(i)}) \leq V_i^t(\bar{\pi}^{(i)})$ for all $t \in \mathbb{N}$). Fourth, it is below the payoff obtained by punishing any other player at any date (i.e., $V_i(\bar{\pi}^{(i)}) \leq V_i^t(\bar{\pi}^{(j)})$ for all $j \in N \setminus \{i\}$ and all $t \in \mathbb{N}$). All these properties are intuitive and well known. In fact, the first guarantees that a player that deviates from the equilibrium payoff is punished regardless of the date of the deviation. The second and third display the typical "stick and carrot" nature of the punishments: players are punished more severely early on. Finally, the fourth properties gives each player an incentive to punish deviators. As we mentioned above, we show that such properties imply that u is a confusion-proof, strictly enforceable subgame perfect equilibrium payoff.

In the case of two players, the structure of the proof is the same. However, since the punishment path needs to the common to both players, it is considerably more difficult task to construct a path $\bar{\pi}$ with the above four properties. In addition, we need to use a mutual minmax action in the initial phase of the punishment path (as in Fudenberg and Maskin (1986)).

Proposition 3 shows that for all $\alpha \in \mathcal{U}^0$, $C(\alpha, \delta)$ is contained in the set of payoffs supported by 1 – memory subgame perfect equilibrium strategies for large δ. We shall now use this result to establish a 1 – memory Folk Theorem result for the set of individually rational payoffs. This is obtained by first establishing that any individually rational payoff vector can be approximated with a confusion-proof payoff profile in $C(\alpha, \delta)$ if δ is sufficiently close to one.

Lemma 3 *Suppose that either* $\dim(\mathcal{U}) = n$ *or* $n = 2$ *and* $\mathcal{U}^0 \neq \emptyset$. *For all* $u \in \mathcal{U}$ *and* $\zeta > 0$ *there exists* $\alpha \in \mathcal{U}^0$ *and* $\tilde{\delta} \in (0, 1)$ *such that for all* $\delta \geq \tilde{\delta}$ *there is* $\tilde{u} \in C(\alpha, \delta)$ *with* $||u - \tilde{u}|| < \zeta$.

Combining Proposition 3 and Lemma 3 we obtain the perfect Folk theorem for 1 – memory strategies.

Theorem 2 *Suppose that either* $\dim(\mathcal{U}) = n$ *or* $n = 2$ *and* $\mathcal{U}^0 \neq \emptyset$. *Then, for all* $u \in \mathcal{U}$ *and* $\zeta > 0$, *there exists* $\delta^* \in (0, 1)$ *such that for all* $\delta \geq \delta^*$, *there is a* 1*–memory subgame perfect equilibrium strategy* f *with* $||U(f) - u|| < \zeta$.

6.3 No Discounting

In this section we assume that players do not discount the future and are interested in the long-term average payoff. The payoff in the supergame $G^\infty(1)$ of G is now given by:

$$U_i^\infty(f) = \liminf_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} u_i(\pi^k(f)).$$

For $\pi \in \Pi$ and $i \in N$, we let $V_i^\infty(\pi) = \liminf_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} u_i(\pi^t)$ be the supergame payoff of player i when the path π is implemented. Finally denote the set of confusion-proof payoffs in the supergame $G^\infty(1)$ that can be obtained through the repetition of a cycle by

$$C = \{u : u = V_i^\infty(\pi) \text{ for some } \pi \in \Pi^c \text{ that consists of a repetition of a cycle}\}.$$

Next, we show that all strictly individually rational payoffs in C can be supported by a $1-$ memory subgame perfect equilibrium.

Proposition 4 *For any payoff profile $u \in \mathcal{U}^0 \cap C$ there exists a 1 – memory subgame perfect equilibrium strategy f with $U^\infty(f) = u$.*

When $n > 2$, the proof of Proposition 4 for the case of a payoff profile $u \in \mathcal{U}^0 \cap C$ involves constructing a confusion proof simple strategy profile $(\pi^{(0)}, ..., \pi^{(n)})$ such that $u = V_i^\infty(\pi^{(0)})$ and for each player $i \in N$ the punishement path $\pi^{(i)}$ involve playing first a finite sequence of action profiles with a payoff approximately close the minmax payoff v_i for player i and then playing the equilibrium path $\pi^{(0)}$. In the case of two players, the punishment phase for the two players are identical and consists of a finite sequence that involve plays that induce payoffs close to the mutual minmax payoffs followed by playing the equilibrium path $\pi^{(0)}$.

Since any strictly individually rational payoff $u \in \mathcal{U}$ can be approximated by payoff profiles in the set $\mathcal{U}^0$, we shall next show, using the previous result, that any individually rational payoff $u \in \mathcal{U}$ can be *approximately* implemented by a $1-$ memory subgame perfect equilibrium strategy profile.[10]

Theorem 3 *Suppose that $\mathcal{U}^0$ is nonempty. Then, for all $u \in \mathcal{U}$ and $\zeta > 0$ there exists a 1–memory subgame perfect equilibrium strategy profile f with $\|U^\infty(f) - u\| < \zeta$.*

[10]By a similar argument as in the proof of Theorem , it can also be shown that any $u \in \text{int}(\mathcal{U}^0) \cap V^\infty(\Pi^{cp})$ can also be implemented *exactly* by a $1-$ memory subgame perfect equilibrium profile. This can be established by first noting that for any such $u \in \text{int}(\mathcal{U}^0) \cap V^\infty(\Pi^{cp})$ there exists another payoff $u' \in \mathcal{U}^0 \cap C$ such that $u \geq u'$. The rest of the proof is similar to that of Theorem 4 except for the punishment phase: we construct a confusion proof simple strategy $(\pi^{(0)}, \ldots, \pi^{(n)})$ such that $u = V^\infty(\pi^{(0)})$ and for each player $i \in N$ the punishement path $\pi^{(i)}$ involve playing first a finite sequence of action profiles with a payoff approximately close the minmax payoff v_i for player i and then playing a single confusion proof path that induces $u' \leq u$.

7 Time dependent Strategies

The notion of a 1 – memory strategy implies that any such strategy must be time independent. In particular, if π is the outcome path that a 1 – memory strategy f induces, then $\pi^{t+1} = \pi^{r+1}$ if $\pi^t = \pi^r$. Thus, either the action profile prescribed in some date never repeats itself, or it will form a loop. These restrictions imply that, in general, not all payoff vectors can be supported by 1 – memory strategies.

One way to avoid this difficulty is to consider time dependent strategies. A strategy $f_i \in F_i$ for player i is a *1 – memory time dependent strategy* if $f_i(h) = f_i(\bar{h})$ for all $h, \bar{h} \in H_t$ satisfying $T(h) = T(\bar{h})$ and all $t \in \mathbb{N}$.

An 1 – memory time dependent strategy allow that $f_i(h)$ differs from $f_i(\bar{h})$ even if $T(h) = T(\bar{h})$. However, this is only possible if the histories have a different length, i.e., if $f_i(h)$ is played in a different time period than $f_i(\bar{h})$. For our purpose, 1 – memory time dependent strategies are appealing because they can support every outcome path and so every payoff vector.

However, we are interested in supporting subgame perfect equilibrium payoffs by 1 – memory time dependent subgame perfect equilibrium strategies. Regarding this goal, similar considerations apply as in Sections 5.

The following is the analog of Theorem 1 for 1 – memory time dependent strategies. The advantage of using time dependent strategies is that u is exactly, and not just approximately, supported by f.

Theorem 4 *Let $\varepsilon \geq 0$ and u be an ε – strictly enforceable subgame perfect equilibrium payoff described by the simple strategy $(\hat{\pi}^{(0)}, \ldots, \hat{\pi}^{(n)})$. Then, there is a 1 – memory contemporaneous ε – subgame perfect equilibrium strategy f such that $U(f) = u$, provided that either $n \geq 3$, or $n = 2$ and $\hat{\pi}^{(1)} = \hat{\pi}^{(2)}$.*

Another advantage of using time dependent strategies appears in our Folk Theorems. In fact, there is no longer the need to focus on confusion-proof payoffs, since any payoff can be supported by a 1 – memory time dependent strategy.

For all $\alpha \in \mathcal{U}^0$ let $\tilde{\Lambda}(\alpha, \delta) \subset \Pi$ be the set of all outcome paths $\pi \in \Pi$ such that

$$V_i^t(\pi, \delta) \geq \alpha_i \text{ for all } i \in N \text{ and } t \in \mathbb{N}.$$

Also, let denote the set of payoff vectors that is supported by the set $\tilde{\Lambda}(\alpha, \delta)$ by

$$\tilde{C}(\alpha, \delta) = \{u \in \mathbb{R}^n : u = V(\pi, \delta) \text{ for some } \pi \in \tilde{\Lambda}(\alpha, \delta)\}.$$

Then, since any payoff can be supported by a 1 – memory time dependent strategy, by exactly the same reasoning as before, we obtain the following version of Proposition 3.

Proposition 5 *Suppose that either* $\dim(\mathcal{U}) = n$ *or* $n = 2$ *and* $\mathcal{U}^0 \neq \emptyset$*. Then for all* $\alpha \in \mathcal{U}^0$*, there exists* $\bar{\delta} \in (0, 1)$ *such that for all* $\delta \geq \bar{\delta}$ *the following holds: for all payoffs* $u \in \tilde{C}(\alpha, \delta)$ *there exists 1 – memory subgame perfect equilibrium strategy* f *with* $U(f, \delta) = u$*.*

Using the above we can now obtain the following exact Folk Theorem result with time dependent strategies.

Theorem 5 *For all payoffs* $u \in \text{int}(\mathcal{U}^0)$*, there exists* $\bar{\delta} \in (0, 1)$ *such that for all* $\delta \geq \bar{\delta}$ *there exists 1 – memory subgame perfect equilibrium strategy* f *with* $U(f, \delta) = u$*.*

A Appendix: Examples

A.1 Robustness of the two player example

We consider a perturbed version of the earlier game, where $\epsilon_1, \epsilon_2, \rho_1, \rho_2$ are sufficiently small and possibly negative:

$1\backslash 2$	a	b
a	$(4 + \epsilon_1, 4 + \epsilon_2)$	$(2 + \rho_1, 5)$
b	$(5, 2 + \rho_2)$	$(0, 0)$

We will prove that there is an open neighborhood of values for $\epsilon_1, \epsilon_2, \rho_1, \rho_2; \delta$, for which the subgame perfect equilibrium payoff of $(4 + \epsilon_1, 4 + \epsilon_2)$ (with full memory) cannot be obtained by 1 – memory subgame perfect strategies.

As before, $s_i \in S_i = [0, 1]$ refers to the action (the probability assigned to a) by player i in the stage game. Note, also that for sufficiently small values of $\epsilon_1, \epsilon_2, \rho_1, \rho_2$, the minmax payoff is $(2+\rho_i)$ for player i, $m^1 = (a, b)$, $m^2 = (b, a)$ and both m^1 and m^2 are Nash equilibria.. The mutual minmax profile is $\bar{m} = (m_1^2, m_2^1) = (b, b)$.

Now suppose the above game is played infintely often. If no there are no restrictions on the memory then the payoff of $(4 + \epsilon_1, 4 + \epsilon_2)$ is subgame perfect: play a at each date with the threat of playing m^i forever if i deviates from (a, a), $i = 1, 2$ (further deviations are ignored). This simple strategy defined above is subgame perfect, provided that δ satisfies the following inequality

$$\delta \geq \frac{1 - \epsilon_i}{3 - \rho_i} \text{ for all } i = 1, 2. \tag{11}$$

To establish our claim suppose that contrary to our claim the payoff of $(4 + \epsilon_1, 4 + \epsilon_2)$ can be supported by a 1– memory subgame perfect equilibrium f. But then there exists functions $g_i : [0, 1]^2 \cup \{h_0\} \to [0, 1]$ for all $i = 1, 2$ such that $f_i(h) = g_i(T(h))$ for all h.

Now we will show that the payoff of $(4 + \epsilon_1, 4 + \epsilon_2)$ can only be obtained by repeating (a, a) forever. Let $p_1 = g_1(h_0)$, $q_1 = g_2(h_0)$, $p_t = g_1(p_{t-1}, q_{t-1})$ and $q_t = g_2(p_{t-1}, q_{t-1})$ for all $t \in \mathbb{N}$. Since $U_1(f) = 4 + \epsilon_1$ and $U_2(f) = 4 + \epsilon_2$, it follows that

$$8+\epsilon_1+\epsilon_2 = U_1(f)+U_2(f) = (1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}((8+\epsilon_1+\epsilon_2)p_t q_t+(7+\rho_1)(p_t(1-q_t))+(7+\rho_2)q_t(1-p_t))). \tag{12}$$

Since $((8 + \epsilon_1 + \epsilon_2)p_t q_t + (7 + \rho_1)(p_t(1 - q_t)) + (7 + \rho_2)q_t(1 - p_t)) \leq 8 + \epsilon_1 + \epsilon_2$ for small values of $\epsilon_1, \epsilon_2, \rho_1, \rho_2$, condition (12) hold only if $p_t = q_t = 1$ for all $t \in \mathbb{N}$. Hence, it follows that $g_i(h_0) = g_i(1, 1) = 1$ for all $i = 1, 2$.

Next, consider a deviation by player 2 to a strategy $\bar{f}_2$ defined by $\bar{f}_2(H_0) = q$ and $\bar{f}_2(h) = 1$ for all $h \in H \setminus H_0$. Then,

$$\begin{aligned} U_2(f_1, \bar{f}_2) &\geq (1 - \delta)((4 + \epsilon_2)q + 5(1 - q)) + (1 - \delta)\delta((4 + \epsilon_2)g_1(1, q) \\ &\quad + (2 + \rho_2)(1 - g_1(1, q))) + (2 + \rho_2)\delta^2 \\ &= (1 - \delta)(5 - q + \epsilon_2 q) + (2 + \rho_2)\delta + \delta(1 - \delta)(2 + \epsilon_2 - \rho_2)g_1(1, q). \end{aligned}$$

Since f is a subgame perfect equilibrium, $4 + \epsilon_2 = U_2(f) \geq U_2(f_1, \bar{f}_2)$ implies

$$g_1(1, q) \leq \frac{(1 - \epsilon_2)q}{(2 + \epsilon_2 - \rho_2)\delta} + \frac{(3 - \rho_2)\delta + \epsilon_2 - 1}{(2 + \epsilon_2 - \rho_2)\delta(1 - \delta)}. \tag{13}$$

Symmetrically, for all p

$$g_2(p, 1) \leq \frac{(1 - \epsilon_1)p}{(2 + \epsilon_1 - \rho_1)\delta} + \frac{(3 - \rho_1)\delta + \epsilon_1 - 1}{(2 + \epsilon_1 - \rho_1)\delta(1 - \delta)}. \tag{14}$$

Consider next the strategy $\bar{f}_1$ for player 1 defined by $\bar{f}_1(h) = 1$ for all $h \in H$. Note that for all $q \in S_2$

$$\begin{aligned} U_1(\bar{f}_1, f_2|(1, q)) &\geq (1 - \delta)((4 + \epsilon_1)g_2(1, q) + (2 + \rho_1)(1 - g_2(1, q))) + (2 + \rho_1)\delta \\ &= (2 + \rho_1) + (1 - \delta)(2 + \epsilon_1 - \rho_1)g_2(1, q). \end{aligned}$$

and

$$U_1(f|(1,q)) \leq (1-\delta)((4+\epsilon_1)g_1(1,q)g_2(1,q)+(2+\rho_1)g_1(1,q)(1-g_2(1,q)) \quad (15)$$
$$+5(1-g_1(1,q))g_2(1,q))+5\delta$$
$$= (1-\delta)((2+\rho_1)g_1(1,q)+5g_2(1,q)-(3-\epsilon_1+\rho_1)g_1(1,q)g_2(1,q))+5\delta$$
$$\leq (1-\delta)((2+\rho_1)g_1(1,q)+5g_2(1,q))+5\delta.$$

Again, since f is subgame perfect we have $U_1(f|(1,q)) \geq U_1(\bar{f}_1, f_2|(1,q))$. This implies that

$$g_2(1,q) \geq -\frac{(2+\rho_1)g_1(1,q)}{(3-\epsilon_1+\rho_1)} + \frac{2+\rho_1-5\delta}{(3-\epsilon_1+\rho_1)(1-\delta)} \text{ for all } q \in S_2. \quad (16)$$

Finally, we use inequalities (13), (14) and (16) to show that player 1 has a profitable deviation from f, which contradicts the fact that f is a subgame perfect equilibrium. To show this, consider player 1 deviating from the equilibrium path by choosing strategy $\tilde{f}_1$ defined by $\tilde{f}_1(h_0)=0$ and $\tilde{f}_1(h)=1$ for all $h \in H \setminus H_0$. Then,

$$U_1(\tilde{f}_1, f_2) \geq 5(1-\delta)+\delta(1-\delta)u_1(1,g_2(0,1))+\delta^2(1-\delta)u_1(1,g_2(1,g_2(0,1)))+(2+\rho_1)\delta^3$$
$$\geq 5(1-\delta)+(2+\rho_1)\delta+\delta^2(1-\delta)(2+\epsilon_1-\rho_1)g_2(1,g_2(0,1)),$$

since $u_1(1,g_2(0,1)) \geq 2+\rho_1$ and $u_1(1,g_2(1,g_2(0,1))) = (2+\epsilon_1-\rho_1)g_2(1,g_2(0,1))+(2+\rho_1)$.

Next, we seek a lower bound on $g_2(1,g_2(0,1))$. Note first that, by (14), we have

$$g_2(0,1) \leq \frac{(3-\rho_1)\delta+\epsilon_1-1}{(2+\epsilon_1-\rho_1)\delta(1-\delta)} := \bar{g}_2(0,1).$$

This, together with (13), implies that

$$g_1(1,g_2(0,1)) \leq \frac{(1-\epsilon_2)g_2(0,1)}{(2+\epsilon_2-\rho_2)\delta} + \frac{(3-\rho_2)\delta+\epsilon_2-1}{(2+\epsilon_2-\rho_2)\delta(1-\delta)}$$
$$\leq \frac{(1-\epsilon_2)\bar{g}_2(0,1)}{(2+\epsilon_2-\rho_2)\delta} + \frac{(3-\rho_2)\delta+\epsilon_2-1}{(2+\epsilon_2-\rho_2)\delta(1-\delta)} := \bar{g}_1(1,\bar{g}_2(0,1)).$$

But then, by (16), we obtain the desired lower bound on $g_2(1,g_2(0,1))$ as follows:

$$g_2(1,g_2(0,1)) \geq -\frac{(2+\rho_1)g_1(1,g_2(0,1))}{(3-\epsilon_1+\rho_1)} + \frac{2+\rho_1-5\delta}{(3-\epsilon_1+\rho_1)(1-\delta)} \geq$$
$$-\frac{(2+\rho_1)\bar{g}_1(1,\bar{g}_2(0,1))}{(3-\epsilon_1+\rho_1)} + \frac{2+\rho_1-5\delta}{(3-\epsilon_1+\rho_1)(1-\delta)} := \bar{g}_2(1,\bar{g}_2(0,1)).$$

Using the lower bound $\bar{g}_2(1, \bar{g}_2(0,1))$ for $g_2(1, g_2(0,1))$, we obtain

$$U_1(\tilde{f}_1, f_2) \geq 5(1-\delta) + (2+\rho_1)\delta + \delta^2(1-\delta)(2+\epsilon_1-\rho_1)\bar{g}_2(1, \bar{g}_2(0,1)).$$

Now as $\delta \to 1/3$, $\epsilon_j \to 0$ and $\rho_j \to 0$ for all $j = 1, 2$, it follows that $\bar{g}_2(0,1) \to 0$, $\bar{g}_1(1, \bar{g}_2(0,1)) \to 0$ and $\bar{g}_2(1, \bar{g}_2(0,1)) \to 1/6$, which imply that

$$5(1-\delta) + (2+\rho_1)\delta + \delta^2(1-\delta)(2+\epsilon_1-\rho_1)\bar{g}_2(1, \bar{g}_2(0,1)) \to 4 + \frac{2}{81}.$$

Therefore, there exists $\phi^* > 0$, $\epsilon_j^* > 0$, and $\rho_j^* > 0$ such that for all $|\delta - 1/3| < \phi^*$, $|\epsilon_j| < \epsilon_j^*$ and $|\rho_j| < \rho_j^*$ we have $U_1(\tilde{f}_1, f_2) > 4 + \epsilon_1$, delivering the required contradiction.

Thus, for all values of $(\delta, \epsilon_1, \epsilon_2, \rho_1, \rho_2)$ which reside in $G \cap B$ where $G = \{(\delta, \epsilon_1, \epsilon_2, \rho_1, \rho_2) : 4+\epsilon_j > 5\delta + (1-\delta)(2+\rho_j), j = 1, 2\}$ and $B = \{(\delta, \epsilon_1, \epsilon_2, \rho_1, \rho_2) : |\delta - 1/3| < \phi^*, |\epsilon_j| < \epsilon_j^* \text{ and } |\rho_j| < \rho_j^*, j = 1, 2\}$, there exists a subgame perfect equilibrium payoff which cannot be obtained by 1 – memory subgame perfect equilibrium.

A.2 Example for the corollary

Consider again the case of $\delta = 1/3$ and $\epsilon_1 = \epsilon_2 = \rho_1 = \rho_2 = 0$. We claim that there exists $\eta > 0$ such that no $u \in B_\eta(4, 4)$ can be supported by a 1 – memory subgame perfect equilibrium.

Let $\eta > 0$ and consider $u \in B_\eta(4, 4)$. Then, $8 - 2\varepsilon < u_1 + u_2 \leq 8$. We claim that if $\pi = \{(p_t, q_t)\}_{t=1}^\infty$ is such that $V(\pi) = u$, then both $p_1 \geq 1 - 3\eta$ and $q_1 \geq 1 - 3\eta$.

Since the problem is symmetric, we deal only with the first case. Let $\gamma = 1 - 3\eta$ and suppose, in order the reach a contradiction, that $p_1 < \gamma$. For all $t \in \mathbb{N}$, let $V_t = V_1^t(\pi) + V_2^t(\pi)$. Then, $V_1 > 8 - 2\eta$, $V_2 \leq 8$ and

$$V_1 = \frac{2}{3}(8p_1q_1 + 7p_1(1-q_1) + 7q_1(1-p_1)) + \frac{V_2}{3} = \frac{2}{3}(7 + p_1q_1 - 7(1-p_1)(1-q_1)) + \frac{V_2}{3} \tag{17}$$

since $p_1(1-q_1) + q_1(1-p_1) = 1 - p_1q_1 - (1-p_1)(1-q_1)$. Since $p_1 < \gamma$, then $p_1q_1 < \gamma q_1$ and $-7(1-p_1) < -7(1-\gamma)$. Thus,

$$8 - 2\eta < \frac{2}{3}(7 + \gamma q_1 - 7(1-\gamma)(1-q_1)) + \frac{V_2}{3} = \frac{2}{3}(-6\gamma q + 7(\gamma + q_1)) + \frac{V_2}{3}. \tag{18}$$

Since $q_1 = 1$ minimizes $12\gamma q_1 - 14q_1$, inequality (18) implies that

$$V_2 > 24 - 6\eta + 12\gamma q_1 - 14(\gamma + q_1) \geq 10 - 6\eta - 2\gamma = 8, \tag{19}$$

a contradiction.

In conclusion, we have that for all $\eta > 0$, $u \in B_\eta(4,4)$ and outcomes π such that $V(\pi) = u$, then $V_i(\pi) > 4 - \eta$ for all $i = 1, 2$, $p_1 \geq 1 - 3\eta$ and $q_1 \geq 1 - 3\eta$. Proceeding in a similar way as above, we can show that if η is sufficiently close to zero, then no 1 – memory strategy supporting u is subgame perfect.

B Appendix: Proofs

Proof of Remark 1. Suppose that $(\pi^{(0)}, \pi^{(1)}, \pi^{(2)})$ is confusion proof. Consider the four possible values that $\Omega(\{1,1\},\{2,1\})$ can take. First, note that it cannot be that $\Omega(\{1,1\},\{2,1\}) = \{1,2\}$; otherwise, this means that $\pi^{(1),1} \neq \pi^{(2),1}$, while by part 3 of Definition 3 we have $\pi^{(1)} = \pi^{(2)}$.

Second, if $\Omega(\{1,1\},\{2,1\}) = \emptyset$ then $\pi^{(1),1} = \pi^{(2),1}$. Proceeding by induction, assume that $\pi^{(1),r} = \pi^{(2),r}$ for all $r = 1, \ldots, t-1$. Then, $\Omega(\{1,t-1\},\{2,t-1\}) = \emptyset$ implies that $\pi^{(1),t} = \pi^{(2),t}$. Hence, $\pi^{(1)} = \pi^{(2)}$.

Suppose that $\Omega(\{1,1\},\{2,1\}) = \{1\}$, which means that $\pi_1^{(1),1} \neq \pi_1^{(2),1}$. By part 2 of Definition 3 we have $\pi^{(1),1} = \pi^{(1),2} = \pi^{(2),2}$. Proceeding by induction as above, one can prove that $\pi^{(1),t} = \pi^{(1),t+1}$ for all $t \in \mathbb{N}$ and that $\pi^{(1),t} = \pi^{(2),t}$ for all $t \geq 2$. This completes the proof, since the remaining case $(\Omega(\{1,1\},\{2,1\}) = \{2\})$ is just analogous to this one. ■

Proof of Proposition 1. (Sufficiency) Let $(\pi^{(0)}, \ldots, \pi^{(n)})$ be a confusion-proof profile of outcome paths. Let $i \in N$ and define f_i as follows: for any $h \in H$, $j \in \{0, \ldots, n\}$, $l \in N$ and $t \in \mathbb{N}$

$$f_i(h) = \begin{cases} \pi_i^{(j),t+1} & \text{if } T(h) = \pi^{(j),t}, \\ \pi_i^{(l),1} & \text{if } T(h) = (s_l, \pi_{-l}^{(j),t}) \text{ and } s_l \neq \pi_l^{(j),t}, \\ \pi_i^{(0),1} & \text{otherwise.} \end{cases}$$

Now we show that f is a well defined function. First, suppose that $\pi^{(j),t} = \pi^{(k),r}$ for some $k, j \in \{0, 1, \ldots, n\}$ and $r, t \in \mathbb{N}$. Then, f is well defined if $\pi^{(k),r+1} = \pi^{(j),t+1}$. Since $(\pi^{(0)}, \ldots, \pi^{(n)})$ is confusion-proof, it follows from part 1 of of Definition 3 that this is indeed the case.

Second, suppose that $\pi^{(k),r} = (s_l, \pi_{-l}^{(j),t})$ and $s_l \neq \pi_l^{(j),t}$ for $k, j \in \{0, 1, \ldots, n\}$, $l \in N$ and $r, t \in \mathbb{N}$. Then, f is well defined only if $\pi^{(k),r+1} = \pi^{(l),1}$. Since $(\pi^{(0)}, \ldots, \pi^{(n)})$ is confusion-proof and $\Omega(\{k,r\},\{j,t\}) = \{l\}$, it follows from part 2 of Definition 3 that this is indeed the case.

Finally, suppose that $(s_l, \pi_{-l}^{(j),t}) = (s_k, \pi_{-k}^{(m),r})$, $s_k \neq \pi_k^{(m),r}$ and $s_l \neq \pi^{(j),t}$ for some $j, m \in$

$\{0, 1, \ldots, n\}$, $k, l \in N$ and $r, t \in \mathbb{N}$. Then f is well defined only if $\pi^{(l),1} = \pi^{(k),1}$. Note that it must be that $s_l = \pi_l^{(m),r}$ and $s_k = \pi_k^{(j),t}$. Hence, $\pi_l^{(m),r} \neq \pi_l^{(j),t}$ and $\pi_k^{(m),r} \neq \pi_k^{(j),t}$, implying that $\Omega(\{m, r\}, \{j, t\}) = \{k, l\}$. Since $(\pi^{(0)}, \ldots, \pi^{(n)})$ is confusion-proof, it follows from part 3 of Definition 3 that $\pi^{(l),1} = \pi^{(k),1}$.

It is clear that the strategy $f = (f_1 \ldots, f_n)$ has 1 – memory, since, by definition, f_i depends only on $T(h)$ for all $i \in N$.

Note, also that f has the following property: $\pi(f) = \pi^{(0)}$ and if player $i \in N$ deviates unilaterally in phase t in any state j, then $\pi^{(i)}$ will be played starting from period $t + 1$. Therefore, f defined by $(\pi^{(0)}, \ldots, \pi^{(n)})$ is a 1 – memory simple strategy.

(Necessity) Let f be a 1 – memory simple strategy represented by $(\pi^{(0)}, \ldots, \pi^{(n)})$. Let $i, j \in \{0, \ldots, n\}$ and $t, r \in \mathbb{N}$.

Suppose that $\Omega(\{i, t\}, \{j, r\}) = \emptyset$. Then, $\pi^{(i),t} = \pi^{(j),r}$. Let $h_1 = (\pi^{(i),t})$ and $h_2 = (\pi^{(j),r})$. Since $T(h_1) = h_1 = h_2 = T(h_2)$ and f has 1 – memory, we have $f(h_1) = f(h_2)$. But then part 1 of Definition 3 is satisfied because $f(h_1) = \pi^{(i),t+1}$ and $f(h_2) = \pi^{(j),r+1}$.

Suppose next that $\Omega(\{i, t\}, \{j, r\}) = \{k\}$ for some $k \in N$. Then, $\pi_l^{(i),t} = \pi_l^{(j),r}$ for all $l \neq k$, while $\pi_k^{(i),t} \neq \pi_k^{(j),r}$. Consider $s_k = \pi_k^{(i),t}$ and $\bar{s}_k = \pi_k^{(j),r}$. Then, $(s_k, \pi_{-k}^{(j),r}) = \pi^{(i),t}$ and since f is a 1 – memory simple strategy, it follows that

$$\pi^{(k),1} = f((s_k, \pi_{-k}^{(j),r})) = f(\pi^{(i),t}) = \pi^{(i),t+1}.$$

Similarly, $(\bar{s}_k, \pi_{-k}^{i),t}) = \pi^{(j),r}$ and so,

$$\pi^{(k),1} = f((\bar{s}_k, \pi_{-k}^{(i),t})) = f(\pi^{(j),r}) = \pi^{(j),r+1}.$$

Hence, $\pi^{(k),1} = \pi^{(j),r+1} = \pi^{(i),t+1}$ and part 2 of Definition 3 is satisfied.

Finally, suppose that $\Omega(\{i, t\}, \{j, r\}) = \{k, l\}$ for some $k, l \in N$. Then, $\pi_m^{(i),t} = \pi_m^{(j),r}$ for all $m \notin \{k, l\}$, while $\pi_k^{(i),t} \neq \pi_k^{(j),r}$ and $\pi_l^{(i),t} \neq \pi_l^{(j),r}$. Consider $s_k = \pi_k^{(j),r}$ and $s_l = \pi_l^{(i),t}$. Then, $(s_l, \pi_{-l}^{(j),r}) = (s_k, \pi_{-k}^{(i),t})$ and since f is a 1 – memory simple strategy, it follows that

$$\pi^{(l),1} = f((s_l, \pi_{-l}^{(j),r})) = f((s_k, \pi_{-k}^{(i),t})) = \pi^{(k),1}.$$

Hence, by induction, $\pi^{(l)} = \pi^{(k)}$ and part 3 of Definition 3 is satisfied. ■

Proof of Theorem 1. Let $\varepsilon \geq 0$, $\eta > 0$ and u be an ε – strictly enforceable subgame perfect equilibrium payoff vector decribed by $(\hat{\pi}^{(0)}, \ldots, \hat{\pi}^{(n)})$. For all $j \in \{0, 1, \ldots, n\}$ and $i \in N$, define

$\zeta_i^{(j)}$ by

$$\zeta_i^{(j)} = \inf_{t\in\mathbb{N}} \left(V_i^t(\hat{\pi}^{(j)}) - \left((1-\delta) \max_{s_i} u_i(s_i, \hat{\pi}_{-i}^{(j),t}) + \delta V_i(\hat{\pi}^{(i)}) \right) \right).$$

Let γ be defined by

$$\gamma = \min \left\{ \eta, \frac{1}{2} \left(\min_{j\in\{0,1,\dots,n\}, i\in N} \{\zeta_i^{(j)}\} + \varepsilon \right) \right\}. \tag{20}$$

It follows that $\gamma > 0$ since $\eta > 0$ and u is an $\varepsilon -$ strictly enforceable subgame perfect equilibrium payoff vector.

Let $\psi > 0$ be such that $d(x,y) < \psi$ implies $|u_i(x) - u_i(y)| < \gamma$ and $|\max_{z_i} u_i(z_i, x_{-i}) - \max_{z_i} u_i(z_i, y_{-i})| < \gamma$, for all $i \in N$. Since S_i is connected for all $i \in N$ it follows that for every $j = 0, 1, \dots n$ and $t \in \mathbb{N}$, $B_\psi(\hat{\pi}^{(j),t}) \cap S$ is uncountable. Thus, we can construct a simple outcome paths $(\bar{\pi}^{(0)}, \bar{\pi}^{(1)}, \dots, \bar{\pi}^{(n)})$ satisfying the conditions described in Lemma 1. Thus, $(\bar{\pi}^{(0)}, \bar{\pi}^{(1)}, \dots, \bar{\pi}^{(n)})$ is confusion proof. Therefore, by Proposition 1, there exists a 1-memory strategy profile f that is represented by it. Moreover, since $\gamma \leq \eta$ we have that $||U_i(f) - u_i|| = ||V_i(\bar{\pi}^{(0)}) - V_i(\hat{\pi}^{(0)})|| < \eta$ for all i.

To complete the proof we need to show f is contemporaneous $\varepsilon -$ subgame perfect. Now fix any $t \in \mathbb{N}$, $i \in N$ and $j \in \{0, 1, \dots, n\}$. Since $V_i^t(\hat{\pi}^{(j)}) - \gamma < V_i^t(\bar{\pi}^{(j)}) < V_i^t(\hat{\pi}^{(j)}) + \gamma$ and $\max_{s_i} u_i(s_i, \bar{\pi}_{-i}^{(j),t}) < \max_{s_i} u_i(s_i, \hat{\pi}_{-i}^{(j),t}) + \gamma$, it follows from (20) that

$$\begin{aligned}
&V_i^t(\bar{\pi}^{(j)}) - (1-\delta) \max_{s_i} u_i(s_i, \bar{\pi}_{-i}^{(j),t}) - \delta V_i(\bar{\pi}^{(i)}) > \\
&V_i^t(\hat{\pi}^{(j)}) - (1-\delta) \max_{s_i} u_i(s_i, \hat{\pi}_{-i}^{(j),t}) - \delta V_i(\hat{\pi}^{(i)}) - 2\gamma \geq \\
&\zeta_i^{(j)} - 2\gamma \geq 2\gamma - \varepsilon - 2\gamma = -\varepsilon.
\end{aligned}$$

Hence, it does not pay player i to deviate from the path induced by state (j) by more than ε. Thus, f is a confusion-proof contemporaneous $\varepsilon -$ subgame perfect. ■

Proof of Lemma 2. For part 1, note that, since $\dim(\mathcal{U}) = n$, then there exists $u \in \text{int}(\mathcal{U})$ (see Theorem 6.2 in Rockafellar (1970)). Let $\varepsilon > 0$ be such that $B_\varepsilon(u) \subseteq \mathcal{U}$. Then, $\bar{u} = (u_i + \varepsilon/2)_i$ is such that $B_{\varepsilon/2}(\bar{u}) \subseteq \mathcal{U}^0$: if $||\tilde{u} - \bar{u}|| < \varepsilon/2$, then $||\tilde{u} - u|| < \varepsilon$ and so $\tilde{u} \in \mathcal{U}$; furthermore, $\tilde{u}_i > \bar{u}_i - \varepsilon/2 = u_i + \varepsilon/2 - \varepsilon/2 = u_i \geq v_i$ and so, in fact, $\tilde{u} \in \mathcal{U}^0$. This implies that $\bar{u} \in \text{int}(\mathcal{U}^0)$.

In order to prove part 2, note first that $D = \text{co}(u(S))$ is compact, since $u(S)$ is compact. It is then clear that $\overline{\mathcal{U}^0} \subseteq \mathcal{U}$ and so $\overline{\text{int}(\mathcal{U}^0)} \subseteq \mathcal{U}$.

Conversely, let $u \in \mathcal{U}$. Let $\bar{u} \in \text{int}(\mathcal{U}^0) \subseteq \text{int}(\mathcal{U})$. Define $u^k = \frac{1}{k}\bar{u} + \left(1 - \frac{1}{k}\right) u$. Then, $u^k \in \mathcal{U}^0$

since $u_i^k > v_i$ for all i and $u^k \in \text{int}(\mathcal{U})$ by Theorems 6.1 in Rockafellar (1970). So $B_\rho(u^k) \subseteq \mathcal{U}^0$ for some $\rho > 0$, i.e., $u^k \in \text{int}(\mathcal{U}^0)$. Moreover, $u^k \to u$. Thus, $u \in \overline{\text{int}(\mathcal{U}^0)}$.

Finally, we prove part 3. Here, we normalize payoffs so that $v_i = 0$ for all $i \in N$.

Let $\alpha \in \mathcal{U}^{0*}$. Since $\dim(\mathcal{U}^0) = n$, then $\text{int}(\mathcal{U}^0)$ is nonempty and so let $u_0 \in \text{int}(\mathcal{U}^0)$. For all $i \in N$, define

$$\tilde{u}^{(i)} = \lambda(\theta\alpha + (1-\theta)g(M^i)) + (1-\lambda)u_0, \tag{21}$$

where $\theta \in (0,1)$ is chosen sufficiently close to 1 so that $\theta\alpha + (1-\theta)g(M^i) \in \mathcal{U}^0$ and $\lambda \in (0,1)$ is chosen sufficiently close to 1 so that $\tilde{u}_i^{(i)} < \alpha_i$ (note that the payoff for player i in $\theta\alpha + (1-\theta)g(M^i)$ is equal to $\theta\alpha_i < \alpha_i$). Since $\theta\alpha + (1-\theta)g(M^i)$ belongs to $\mathcal{U}^0$, it follows from Rockafellar (1970, Theorem 6.1, p. 45) that $\tilde{u}^{(i)} \in \text{int}(\mathcal{U}^0)$. Also, let $\varepsilon > 0$ be such that $B_{2\varepsilon}(\tilde{u}^{(i)}) \subseteq V^*$ for all $i \in N$.

Let $i \in N$ and let $k(i) \in N$ be such that $\tilde{u}_i^{(k(i))} \leq \tilde{u}_i^{(l)}$ for all $l \in N$. Then, define

$$y^i = \tilde{u}^{(k(i))} - \varepsilon e_i. \tag{22}$$

Since $||y^{(i)} - \tilde{u}^{(i)}|| \leq \varepsilon$, it follows that $u^{(i)} \in \mathcal{U}^0$ for all $i \in N$. Furthermore, we have $y_i^i = \tilde{u}_i^{(k(i))} - \varepsilon \leq \tilde{u}_i^{(i)} - \varepsilon < \alpha_i$, for all $i \in N$. Finally, we claim that $y_i^i < y_i^j$ for all $i, j \in N$ such that $j \neq i$. Indeed,

$$y_i^i = \tilde{u}_i^{(k(i))} - \varepsilon \leq \tilde{u}_i^{(k(j))} - \varepsilon = y_i^j - \varepsilon < y_i^j.$$

■

Proof of Proposition 3. We consider the two cases of $n \geq 3$ (**Case A**) and the $n = 2$ (**Case B**) separately.

Case A. $n \geq 3$ and $\dim(\mathcal{U}) = n$.

For convenience, in this case we normalize payoffs so that $v_i = 0$ for all $i \in N$.

Fix any $\alpha \in \mathcal{U}^0$. Then, by Lemma 3, for all $i \in N$ there exists $y^i \in \mathcal{U}^0$ satisfying the following property: for all $i, j \in N$ with $j \neq i$, $y_i^i < \alpha_i$ and $y_i^i < y_i^j$.

Define γ' to be such that $2\gamma' = \min\{y_i^j, \alpha_i\} - y_i^i$. Let $\alpha'_i = y_i^i + \gamma'$ for each i. Clearly, $\gamma' < \alpha'_i < \alpha_i$. Also, by the properties of $(y^1, \ldots, y^n)$ we have $\gamma' > 0$ for all i.

Next, let $\gamma = \min\{\gamma', y_1^1, ..., y_n^n\}$. Also, suppose $T \in \mathbb{N}$ is such that

$$T \geq \frac{4M}{\gamma} = \frac{M}{\gamma/4}, \tag{23}$$

where $M = \max_i \max_{s \in S} |u_i(s)|$.

Denote D_k to be the set of achievable payoff in the finite game that consists of repeating the one-shot game k times, and in which payoffs consist of the average of the payoffs obtained in the k stages. Also, let $K \in \mathbb{N}$ be such that $D \subseteq \cup_{x \in D_K} B_{\gamma/2}(x)$ (see Sorin (1992, Proposition 1.3.)). Finally, denote $\bar{\delta} \in (0,1)$ to be such that $\delta \geq \bar{\delta}$ implies

$$\frac{1-\delta^{T+1}}{1-\delta} > T, \tag{24}$$

$$\frac{\delta^T}{1-\delta^T} > \frac{8M}{\gamma} = \frac{2M}{\gamma/4} \text{ and} \tag{25}$$

$$\sup_{x \in [-M-\gamma, M+\gamma]^K} \left| \frac{1-\delta}{1-\delta^K} \sum_{k=1}^{K} \delta^{k-1} x_k - \frac{1}{K} \sum_{k=1}^{K} x_k \right| < \frac{\gamma}{4}. \tag{26}$$

Now fix any $\delta \geq \bar{\delta}$ and consider any $u \in C(\alpha, \delta)$. We will show that there is a 1–memory subgame perfect equilibrium strategy profile f with $U(f, \delta) = u$.

Let π be a confusion proof path such that $u_i = V_i(\pi, \delta)$ and $V_i^t(\pi, \delta) \geq \alpha_i$ for all $i \in N$ and $t \in \mathbb{N}$.

Also, let $x^i \in D_K$ be such that $\|x^i - y^i\| < \gamma/2$ for all $i \in N$. Then, by the definitions of y^i, α_i' and γ, the following hold for all i:

$$x_i^i < y_i^i + \frac{\gamma}{2} = \alpha_i' - \gamma' + \frac{\gamma}{2} \leq \alpha_i' - \frac{\gamma}{2}; \tag{27}$$

$$x_i^i > y_i^i - \frac{\gamma}{2} \geq \gamma - \frac{\gamma}{2} = \frac{\gamma}{2}; \tag{28}$$

$$x_i^j > y_i^j - \frac{\gamma}{2} \geq y_i^i + 2\gamma' - \frac{\gamma}{2} = \alpha_i' + \gamma' - \frac{\gamma}{2} \geq \alpha_i' + \frac{\gamma}{2}. \tag{29}$$

Next, let $\{s^{i,k}\}_{k=1}^K$ be such that

$$\frac{1}{K} \sum_{k=1}^{K} u_j(s^{i,k}) = x_j^i,$$

for all $i, j \in N$. Consider

$$\beta_i^{(t)} = \frac{1-\delta}{1-\delta^K} \left[\sum_{k=t}^{K} \delta^{k-t} u_i(s^{i,k}) + \sum_{k=1}^{t-1} \delta^{k+T-t} u_i(s^{i,k}) \right],$$

and let t^* be such that $\beta_i^{(t^*)} \leq \beta_i^{(t)}$ for all $1 \leq t \leq K$. Denote $\beta_i^{(t^*)}$ by β_i and let $\bar{\pi}^{(i)}$ consist of repetitions of $(s^{i,t^*}, s^{i,t^*+1}, \ldots, s^{i,K}, s^{i,1}, \ldots, s^{i,t^*-1})$. Note that $V_i(\bar{\pi}^{(i)}) = \beta_i$. Furthermore, using

(26), it follows, respectively, from (27), (28) and (29) that for all i

$$\beta_i < x_i^i + \frac{\gamma}{4} < \alpha_i' - \frac{\gamma}{2} + \frac{\gamma}{4} < \alpha_i', \tag{30}$$

$$\beta_i > x_i^i - \frac{\gamma}{4} > \frac{\gamma}{2} - \frac{\gamma}{4} = \frac{\gamma}{4} \tag{31}$$

$$\text{and } V_i^t(\bar{\pi}^{(j)}) > x_i^j - \frac{\gamma}{4} > \alpha_i' + \frac{\gamma}{2} - \frac{\gamma}{4} = \alpha_i' + \frac{\gamma}{4} > \beta_i. \tag{32}$$

Let $\pi^{(0)} = \pi$ and define outcomes $\pi^{(i)}$ by

$$\pi^{(i),t} = \begin{cases} m^i & \text{if } t < T, \\ \bar{\pi}^{(i),t-T+1} & \text{if } t \geq T. \end{cases}$$

We claim that the simple strategy defined by $(\pi^{(0)}, \ldots, \pi^{(n)})$ supports u as a strictly enforceable subgame perfect equilibrium payoff.

Let $i \in N$ be given. First, consider player i deviating from the equilibrium path $\pi^{(0)} = \pi$ in period t. Then we have

$$V_i^t(\pi) - \left((1-\delta)\max_{s_i} u_i(s_i, \pi_{-i}^t) + \delta V_i(\pi^{(i)})\right) = \tag{33}$$

$$V_i^t(\pi) - \left((1-\delta)\max_{s_i} u_i(s_i, \pi_{-i}^t) + \delta^{T+1}\beta_i\right) > \tag{34}$$

$$\alpha_i' - \left((1-\delta)M + \delta^{T+1}\alpha_i'\right). \tag{35}$$

(The last inequality follows from $V_i^t(\pi) \geq \alpha_i > \alpha_i'$ and from (30).) But then since $\alpha_i' > (1-\delta)M + \delta^{T+1}\alpha_i'$ is equivalent to

$$\frac{1-\delta^{T+1}}{1-\delta} > \frac{M}{\alpha_i'},$$

and, by (24), (23) and the definition of γ,

$$\frac{1-\delta^{T+1}}{1-\delta} > T \geq \frac{M}{\gamma/4} \geq \frac{M}{\alpha_i'},$$

it follows that

$$\inf_t \left[V_i^t(\pi) - \left((1-\delta)\max_{s_i} u_i(s_i, \pi_{-i}^{(0),t}) + \delta V_i(\pi^{(i)})\right)\right] > 0.$$

Second, consider a deviation from $\pi^{(i),t}$. If $t < T$ we have that

$$V_i^t(\pi^{(i)}) - \left((1-\delta)\max_{s_i} u_i(s_i, m_{-i}^i) + \delta V_i(\pi^{(i)})\right) = \tag{36}$$

$$\delta^{T-t}\beta_i - \delta^{T+1}\beta_i > 0. \tag{37}$$

(this follows from $\max_{s_i} u_i(s_i, m^i_{-i}) = v_i = 0$ and $\beta_i > 0$). If $t \geq T$ we have that

$$V_i^t(\pi^{(i)}) - \left((1-\delta) \max_{s_i} u_i(s_i, \pi_{-i}^{(i),t}) + \delta V_i(\pi^{(i)}) \right) = \tag{38}$$

$$\beta_i^{(t-T+1 \bmod K)} - \left((1-\delta) \max_{s_i} u_i(s_i, \pi_{-i}^{(i),t}) + \delta^{T+1}\beta_i \right) \geq \tag{39}$$

$$\beta_i - \left((1-\delta)M + \delta^{T+1}\beta_i \right). \tag{40}$$

Since $\beta_i > (1-\delta)M + \delta^{T+1}\beta_i$ is equivalent to

$$\frac{1-\delta^{T+1}}{1-\delta} > \frac{M}{\beta_i},$$

and, by (23), (24) and (31),

$$\frac{1-\delta^{T+1}}{1-\delta} > T \geq \frac{M}{\gamma/4} > \frac{M}{\beta_i},$$

it follows that

$$\inf_t \left[V_i^t(\pi^{(i)}) - \left((1-\delta) \max_{s_i} u_i(s_i, \pi_{-i}^{(i),t}) + \delta V_i(\pi^{(i)}) \right) \right] > 0.$$

Finally, consider a deviation from $\pi^{(j),t}$, $j \neq i$. If $t \geq T$, then since $V_i(\pi^{(i)}) = \delta^{T+1}\beta_i$ it follows from (32) that

$$V_i^t(\pi^{(j)}) - \left((1-\delta) \max_{s_i} u_i(s_i, m^j_{-i}) + \delta V_i(\pi^{(i)}) \right) = \tag{41}$$

$$V_i^{t-T}(\bar{\pi}^{(j)}) - \left((1-\delta) \max_{s_i} u_i(s_i, m^j_{-i}) + \delta^{T+1}\beta_i \right) > \tag{42}$$

$$\beta_i - \left((1-\delta)M + \delta^{T+1}\beta_i \right) > 0. \tag{43}$$

If $t < T$, then again using (30) and (32) we have

$$\begin{aligned}
& V_i^t(\pi^{(j)}) - \left((1-\delta) \max_{s_i} u_i(s_i, m^j_{-i}) + \delta V_i(\pi^{(i)}) \right) = \\
& (1-\delta^{T-t}) u_i(m^j) + \delta^{T-t} V_i(\bar{\pi}^{(j)}) - \left((1-\delta) \max_{s_i} u_i(s_i, m^j_{-i}) + \delta^{T+1}\beta_i \right) > \\
& -(1-\delta^{T-t})M + \delta^{T-t}(\alpha_i' + \frac{\gamma}{4}) - \left((1-\delta)M + \delta^{T+1}\alpha_i' \right) > \\
& -(1-\delta^{T})M + \delta^{T}(\alpha_i' + \frac{\gamma}{4}) - \left((1-\delta)M + \delta^{T+1}\alpha_i' \right) > \\
& -(1-\delta^{T})M + \delta^{T}\frac{\gamma}{4} - (1-\delta)M.
\end{aligned}$$

(The last inequality in the above expression follows from $\alpha_i' > 0$.) But since $\frac{\delta^T}{1-\delta^T} > \frac{2M}{\frac{\gamma}{4}}$,then

$$\delta^T \frac{\gamma}{4} > 2(1-\delta^T)M > (1-\delta)M + (1-\delta^T)M.$$

Thus,

$$\inf_t \left[V^t(\pi^{(j)}) - \left((1-\delta) \max_{s_i} u_i(s_i, \pi_{-i}^{(j),t}) + \delta V_i(\pi^{(i)}) \right) \right] > 0.$$

This concludes the proof that the simple strategy described by $(\pi, \pi^{(1)}, \ldots, \pi^{(n)})$ above is a strictly enforceable and induces a payoff of u. But then since by assumption π is a confusion-proof single path, by Proposition 2, there exists a 1 – memory subgame perfect equilibrium strategy f with $U(f) = u$. ■

Case B. $n = 2$.

Let $\bar{m} = (m_1^2, m_2^1)$ be the mutual minmax profile. Clearly, $v_i \geq u_i(\bar{m})$.

For convenience, in this case we normalize payoffs so that $u_i(\bar{m}) = 0$ for both $i = 1, 2$.

Let $\alpha \in \mathcal{U}^0$. Consider any $y \in \mathcal{U}^0$ such that $\alpha_i > y_i > v_i$. Since $\mathcal{U}^0$ is convex, $\alpha_i > v_i \geq u_i(\bar{m}) = 0$ for all i, such y exists.

Define $\epsilon_i = \min\{\alpha_i - y_i, y_i - v_i\} > 0$ and $\gamma = \min_i \varepsilon_i/4$. Fix any $\xi > 0$ such that

$$\xi < \min\left\{\gamma, \frac{\gamma^2}{2M}\right\}, \tag{44}$$

where, as before, $M = \max_{i=1,2} \max_s |u_i(s)|$. Clearly, the following two conditions hold:

$$y_i - \xi > v_i + 3\gamma, \text{ for all } i \tag{45}$$

$$y_i + \xi < \alpha_i, \text{ for all } i. \tag{46}$$

Let $K \in \mathbb{N}$ be such that $D \subseteq \cup_{x \in D(K)} B_{\xi/2}(x)$ (see Sorin (1992, Proposition 1.3.)), T be such that

$$T > \frac{M}{\gamma} \tag{47}$$

and $\bar{\delta} \in (0,1)$ be such that for all $\delta \in [\bar{\delta}, 1)$

$$\frac{1-\delta^{T+1}}{1-\delta} > T, \tag{48}$$

$$\delta^T > \frac{M}{M+\gamma}, \tag{49}$$

$$1 - \delta < \frac{\gamma}{2M} \text{ and} \tag{50}$$

$$\sup_{x \in [-M,M]^K} \left| \frac{1-\delta}{1-\delta^K} \sum_{k=1}^K \delta^{k-1} x_k - \frac{1}{K} \sum_{k=1}^K x_k \right| < \frac{\xi}{2}. \tag{51}$$

Now fix any $\delta \geq \bar{\delta}$ and consider any $u \in C(\alpha, \delta)$. We will show that there is a 1–memory subgame perfect equilibrium strategy f with $U(f, \delta) = u$.

Let $\pi \in \Lambda(\alpha, \delta)$ be such that it satisfies $V_i(\pi, \delta) = u_i$ for all $i \in N$.

By (51), let $x \in D(K)$ be such that $\|x - y\| < \xi/2$. Let $\{s^k\}_{k=1}^K$ be such that

$$\frac{1}{K}\sum_{k=1}^{K} u_i(s^k) = x_i$$

for all $i \in N$. Then, let $\tilde{\pi}$ consist of repetitions of $(s^1, \ldots, s^K)$ and let

$$z_i = V_i(\tilde{\pi})$$

for all $i \in N$. Note that $\|x - V^t(\tilde{\pi})\| < \xi/2$ for all $t \in \mathbb{N}$. In particular, we have that $\|x - z\| < \xi/2$ and $\|z - V^t(\tilde{\pi})\| \leq \|z - x\| + \|x - V^t(\tilde{\pi})\| < \xi$ and so

$$V_i^t(\tilde{\pi}) > z_i - \xi, \text{ for all } t \in \mathbb{N}. \tag{52}$$

Furthermore, it follows that

$$\alpha_i > z_i > v_i + 2\gamma, \text{ for all } i. \tag{53}$$

The first inequality in (53) follows from $z_i = V_i(\tilde{\pi}) < x_i + \frac{\xi}{2} < y_i + \xi$ and condition (46), and the second follows from $z_i > x_i - \xi > x_i - \gamma > y_i - \xi - \gamma$ and condition (45).

Next define the common punishment path $\bar{\pi}$ by

$$\bar{\pi}^t = \begin{cases} \bar{m} & \text{if } t < T + R, \\ \tilde{\pi}^{t-T-R+1} & \text{if } t \geq T + R \end{cases}$$

for some $R \in \mathbb{N}$ satisfying

$$v_i + \gamma < \delta^R z_i < z_i - \xi, \text{ for all } i = 1, 2. \tag{54}$$

Before proceeding further with the construction of equilibrium strategy, we shall next establish the existence of a number $R \in \mathbb{N}$ with the above property in the following claim.

Claim 1 *There exists $R \in \mathbb{N}$ such that $v_i + \gamma < \delta^R z_i < z_i - \xi$, for all $i = 1, 2$.*

Proof of Claim 1. Let

$$a = \max_i \frac{v_i + \gamma}{z_i}, \quad b = 1 - \frac{\xi}{\min_i v_i + \gamma} \quad \text{and } l = b - a.$$

Then, it follows that

$$l = 1 - \max_i \frac{v_i+\gamma}{z_i} - \frac{\xi}{\min_i v_i + \gamma} > \frac{\gamma}{M} - \frac{\xi}{\min_i v_i+\gamma} \geq$$
$$\frac{\gamma}{M} - \frac{\xi}{\gamma} \geq \frac{\gamma}{M} - \frac{\gamma}{2M} = \frac{\gamma}{2M} > 0.$$

The first and the third inequality in the above follow respectively from $z_i > v_i + 2\gamma$ (by (53)) and $\xi < \frac{\gamma^2}{2M}$ (by the definition of ξ). This, together with $\frac{\gamma}{2M} > 1-\delta$ (condition (50)), imply that

$$l > 1-\delta \geq \delta^r(1-\delta) = \delta^r - \delta^{r+1} \text{ for all } \; r \in \mathbb{N}_0. \tag{55}$$

Next, we show that there exists $R \in \mathbb{N}$ such that $\delta^R \in (a,b)$. First, note that $\delta^0 = 1 > b$ and $\delta^L < a$ for L sufficiently large. Let r be the smallest integer in $\mathbb{N}_0$ such that $\delta^r \geq b$. Then $\delta^{r+1} < b$, by definition. Moreover, $\delta^{r+1} > a$; otherwise, $\delta^r - \delta^{r+1} \geq b - a = l$, but this contradicts (55). Thus, $\delta^{r+1} \in (a,b)$.

Now since $\delta^R \in (a,b)$, it then follows that for any $i = 1,2$

$$\frac{v_i+\gamma}{z_i} \leq \max_j \frac{v_j+\gamma}{z_j} < \delta^R < 1 - \frac{\xi}{\min_j v_j+\gamma} \leq 1 - \frac{\xi}{v_i+\gamma}.$$

Hence, $v_i + \gamma < \delta^R z_i$. Also, since by (53) $z_i > v_i + 2\gamma$, it follows that $\delta^R < 1 - \frac{\xi}{z_i}$; so $\delta^R z_i < z_i - \xi$.
■

Now let $\pi^{(0)} = \pi$ and $\pi^{(i)} = \bar{\pi}$ for all $i = 1,2$. We claim that the simple strategy $(\pi^{(0)}, \pi^{(1)}, \pi^{(2)})$ supports u as a strictly enforceable subgame perfect equilibrium payoff.

Fix any $i \in 1,2$. Let $\beta_i = V_i^T(\bar{\pi}) = \delta^R z_i = \delta^R V_i(\tilde{\pi})$. We have from (54) and $\alpha_i > z_i$ (by (53)) that

$$\alpha_i > \beta_i > v_i + \gamma \tag{56}$$

Also, since $\delta^R z_i < z_i - \xi < V_i^k(\tilde{\pi})$ for all $k \in \mathbb{N}$, $\delta^R z_i \leq \delta^{T+R-t} z_i$ for all $T \leq t \leq T+R$ and

$$V_i^t(\bar{\pi}) = \begin{cases} \delta^{T+R-t} z_i & \text{if } T \leq t < T+R, \\ V_i^{t-T-R+1}(\tilde{\pi}) & \text{if } t \geq T+R, \end{cases}$$

it follows that

$$\beta_i \leq V_i^t(\bar{\pi}) \text{ for all } t \geq T. \tag{57}$$

Next, consider player i deviating from the equilibrium path $\pi^{(0)} = \pi$ in period t: we have that

$$V_i^t(\pi) - \left((1-\delta)\max_{s_i} u_i(s_i, \pi_{-i}^t) + \delta V_i(\bar{\pi})\right) = \tag{58}$$

$$V_i^t(\pi) - \left((1-\delta)\max_{s_i} u_i(s_i, \pi_{-i}^t) + \delta^{T+1}\beta_i\right) > \tag{59}$$

$$\alpha_i - \left((1-\delta)M + \delta^{T+1}\alpha_i\right). \tag{60}$$

(The equality in the above follows from $V_i(\bar{\pi}) = \delta^{R+T} z_i = \delta^T \beta_i$, the inequality from $\pi \in \Lambda(\alpha, \delta)$ and (56).) Since $\alpha_i > (1-\delta)M + \delta^{T+1}\alpha_i$ is equivalent to $\frac{1-\delta^{T+1}}{1-\delta} > \frac{M}{\alpha_i}$, and

$$\frac{1-\delta^{T+1}}{1-\delta} > T > \frac{M}{\gamma} \geq \frac{M}{\alpha_i}$$

(which holds by (48), (47) and since $\alpha_i - v_i \geq \alpha_i - y_i > \gamma$), it follows that

$$\inf_t \left[V_i^t(\pi) - \left((1-\delta)\max_{s_i} u_i(s_i, \pi_{-i}^t) + \delta V_i(\bar{\pi})\right)\right] > 0.$$

Second, consider a deviation by i from $\bar{\pi}^t$. If $t < T$ we have that

$$V_i^t(\bar{\pi}) - \left((1-\delta)\max_{s_i} u_i(s_i, m_{-i}^i) + \delta V_i(\bar{\pi})\right) = \tag{61}$$

$$\delta^{T-t}\beta_i - \left((1-\delta)v_i + \delta^{T+1}\beta_i\right) \geq \tag{62}$$

$$\delta^T\beta_i - \left((1-\delta)v_i + \delta^{T+1}\beta_i\right), \tag{63}$$

Since, by (49) and (56), $\delta^T > \frac{M}{M+\gamma} \geq \frac{v_i}{v_i+\gamma} > \frac{v_i}{\beta_i}$, then it follows that

$$\delta^T\beta_i - (1-\delta)v_i + \delta^{T+1}\beta_i > 0.$$

If $t \geq T$ we have from (57) that

$$V_i^t(\bar{\pi}) - \left((1-\delta)\max_{s_i} u_i(s_i, \bar{\pi}_{-i}^t) + \delta V_i(\bar{\pi})\right) \geq \tag{64}$$

$$\beta_i - \left((1-\delta)M + \delta^{T+1}\beta_i\right). \tag{65}$$

Since $\beta_i > (1-\delta)M + \delta^{T+1}\beta_i$ is equivalent to $\frac{1-\delta^{T+1}}{1-\delta} > \frac{M}{\beta_i}$,and

$$\frac{1-\delta^{T+1}}{1-\delta} > T > \frac{M}{\gamma} > \frac{M}{\beta_i},$$

(which holds which holds by (48), (47) and (56)), it follows that

$$\inf_t \left[V_i^t(\bar{\pi}) - \left((1-\delta)\max_{s_i} u_i(s_i, \bar{\pi}_{-i}^t) + \delta V_i(\bar{\pi})\right)\right] > 0.$$

This concludes the proof that the simple strategy described above by $(\pi^{(0)}, \pi^{(1)}, \pi^{(2)})$ with $\pi^{(0)} = \pi$ and $\pi^{(1)} = \pi^{(2)} = \overline{\pi}$ is strictly enforceable and induces a payoff of u. But then since by assumption π is a confusion-proof single path, by Proposition 2, there exists a $1-$memory subgame perfect equilibrium strategy f with $U(f) = u$. ■

This completes the proof of Proposition 3. ■

Proof of Lemma 3. Fix any $u \in \mathcal{U}$ and any $\zeta > 0$. Now we make two claims.

Claim 2 *There exists $y, \alpha \in int(U^0)$ and $\xi \in (0, \frac{\zeta}{4})$ such that $||y - u|| < \zeta/4$ and $y_i - 3\xi > \alpha_i$ for all i.*

We shall first prove Claim 2 assuming that $\dim(\mathcal{U}) = n$. Then by Lemma 2, there exists $y \in \text{int}(\mathcal{U}^0)$ such that $||y - u|| < \zeta/4$. Let $\rho > 0$ be such that $B_\rho(y) \subseteq \text{int}(\mathcal{U}^0)$ and define

$$\xi < \min\left\{\frac{\zeta}{4}, \frac{\rho}{6}\right\}. \tag{66}$$

Now consider α defined by $\alpha_i = y_i - \rho/2$ for all $i \in N$. Clearly, $\alpha \in B_\rho(y)$; thus, $\alpha \in \text{int}(\mathcal{U}^0)$. Moreover, since $\xi < \rho/6$, we have

$$y_i - 3\xi = \alpha_i + \rho/2 - 3\xi > \alpha_i, \text{ for all } i.$$

Next, we shall prove Claim 2 assuming that $n = 2$ and $\mathcal{U}^0 \neq \emptyset$. Then there exists $\overline{u} \in \mathcal{U}^0$ such that $\overline{u}_i > v_i$, for all i. This, together with $u_i \geq v_i$, imply that there exists $y \in \mathcal{U}^0$ such that $||y - u|| < \zeta/4$ (take an appropriate convex combination of $\overline{u}$ and u). Next, let $\xi > 0$ be such that $y_i - v_i > 4\xi$. Also, since $y_i > v_i \geq u_i(\bar{m})$ for all i, there exists $\lambda \in (0,1)$ such that $\lambda y + (1-\lambda)u(\bar{m}) \in U^0$. Denote $\lambda y + (1-\lambda)u(\bar{m})$ by α. Thus, $\alpha \in \mathcal{U}^0$ and $y > \alpha$. To complete the proof, define $\xi > 0$ to be such that $\xi < \zeta/4$ and $3\xi < y_i - \alpha_i$ for all i.

Claim 3 *For any $y \in U^0$ and $\xi > 0$, there exists $\tilde{\delta} \in (0,1)$ and a confusion proof single path $\tilde{\pi}$ such that $\|V^t(\tilde{\pi}) - y\| < 3\xi$ for all $t \in N$ and for all $\delta \geq \tilde{\delta}$.*

In order to prove Claim 3, fix any $y \in \mathcal{U}^0$ and $\xi > 0$. Let $K \in \mathbb{N}$ be such that $D \subseteq \cup_{x \in D_K} B_\xi(x)$ and $\tilde{\delta} \in (0,1)$ be such that

$$\sup_{x \in [-M,M]^K} \left| \frac{1-\delta}{1-\delta^K} \sum_{k=1}^{K} \delta^{k-1} x_k - \frac{1}{K} \sum_{k=1}^{K} x_k \right| < \xi, \tag{67}$$

for all $\delta \geq \tilde{\delta}$.

Let $\delta \geq \tilde{\delta}$. Let $\{\bar{s}^k\}_{k=1}^K$ be such that

$$\left\|\frac{1}{K}\sum_{k=1}^{K} u(\bar{s}^k) - y\right\| < \xi.$$

Since S_i is connected for all i, there exist a (finite) sequence $\{s^k\}_{k=1}^K$ such that $|u_i(s^t) - u_i(\bar{s}^t)| < \xi$ for all i and t and $s_i^t \neq s_j^r$ for all $1 \leq t, r \leq K$ and $1 \leq i, j \leq n$ satisfying $(i,t) \neq (j,r)$. Then,

$$\left\|\frac{1}{K}\sum_{k=1}^{K} u(s^k) - \frac{1}{K}\sum_{k=1}^{K} u(\bar{s}^k)\right\| < \xi$$

and so

$$\left\|\frac{1}{K}\sum_{k=1}^{K} u(s^k) - y\right\| < 2\xi.$$

Finally, let $\tilde{\pi}$ be the repetition of $\{s^k\}$. Then we have $||V^t(\tilde{\pi}) - y|| < 3\xi$ for all $t \in \mathbb{N}$. This completes the proof of Claim 3.

Now to complete the proof of Lemma 3, note that it follows from the previous two claims that there exists $y, \alpha \in \mathcal{U}^0$, $\xi < \frac{\zeta}{4}$, $\tilde{\delta} \in (0,1)$ and a confusion proof single path $\tilde{\pi}$ such that $||y - u|| < \zeta/4, y_i - 3\xi > \alpha_i$ for all i and $\|V^t(\tilde{\pi}) - y\| < 3\xi$ for all i and $t \in \mathbb{N}$. Now let $\tilde{u} = V(\tilde{\pi})$. Then $||\tilde{u} - u|| \leq ||u - y|| + ||\tilde{u} - y|| < \zeta/4 + 3\zeta/4 = \zeta$ and $V_i^t(\tilde{\pi}) > y_i - 3\xi > \alpha_i$ for all i and $t \in \mathbb{N}$. But then $\tilde{u} \in C(\alpha, \delta)$.

This completes the proof of Lemma 3. ■

Proof of Proposition 4. Let $u \in \mathcal{U}^0 \cap C$. Then there exists $\varepsilon > 0$ be such that

$$u_i > v_i + \varepsilon \text{ for all } i \in N \tag{68}$$

Moreover, since $u \in \mathcal{U}^0 \cap C$ there exists a confusion proof path $\pi^{(0)}$ that consists of repeatedly playing a finite sequence of action profiles$\{\bar{s}^k\}_{k=1}^C$ such that $u = \frac{1}{C}\sum_{k=1}^C u(\bar{s}^k)$ for all i.

Let $\nu_i : S \rightarrow \mathbb{R}$ be defined by

$$\nu_i(s) = \max_{\bar{s}_i \in S_i} u_i(\bar{s}_i, s_{-i}).$$

Also, define $M_i = \max_{s \in S} |u_i(s)|$. Then, let $R \in \mathbb{N}$ be such that

$$\frac{R\varepsilon}{2} > (M_i - u_i)C, \text{ for all } i \in N, \tag{69}$$

Since S is connected and ν_i is continuous (by the continuity of u_i and compactness of S_i), then $\nu_i(S) \subset \mathbb{R}$ is also connected (it is a closed interval). Connectedness of S and $\nu_i(S)$ together with

(68) (note that $v_i = \nu_i(m^i)$) imply that for each $i \in N$ there exists a set $(b^{(i),t})_{t=1}^R \subset S$ such that

$$b_l^{(i),r} \neq \pi_l^{(0),t} \text{ for all } 1 \leq i, l \leq n, t \in \mathbb{N} \text{ and } 1 \leq r \leq R, \tag{70}$$

$$b_l^{(i),r} \neq b_l^{(j),q} \text{ for all } 1 \leq i, j, l \leq n, \text{ and } 1 \leq r, q \leq R \text{ such that } (i, r) \neq (j, q) \tag{71}$$

$$u_i \geq \nu_i(b^{(i),r}) + \frac{\varepsilon}{2} \text{ for all } 1 \leq r \leq R, \tag{72}$$

and

$$\text{if } n = 2 \text{ then } b^{(i)} = b^{(j)}. \tag{73}$$

By connectedness of S and continuity of v_i, the above four conditions hold because for $n > 2$ for each i there exists a continnum of actions profiles $B^i \subset A$ such that $||\nu_i(b) - v_i|| < \varepsilon/2$ for all $b \in B^i$, and when $n = 2$ there exists a continnum of actions profiles B such that $||\nu_i(b) - \pi_i(m_1^2, m_2^1)|| < \varepsilon/2$, for all $i = 1, 2$, where as before (m_1^2, m_2^1) is the mutual minmax strategies.

We next show that

$$\frac{1}{q+R}\left[\nu_i(\pi^{(0),q}) + \sum_{r=1}^{q-1} u_i(\pi^{(0),r}) + \sum_{r=1}^{R} u_i(b^{(i),r})\right] \leq u_i, \tag{74}$$

for all $1 \leq i \leq n$ and $q \in \mathbb{N}$. To show this let $q = m + d$ with $m, d \in \mathbb{N}_0$, m is a multiple of C and $0 \leq d < C$. Then, by condition (72) and (69), we have

$$\begin{aligned}
&\frac{1}{q+R}\left[\nu_i(\pi^{(0),q}) + \sum_{r=1}^{q-1} u_i(\pi^{(0),r}) + \sum_{r=1}^{R} u_i(b^{(i),r})\right] = \\
&\frac{1}{m+d+R}\left[\nu_i(\pi^{(0),q}) + mu_i + \sum_{r=1}^{d-1} u_i(\pi^{(0),r}) + \sum_{r=1}^{R} u_i(b^{(i),r})\right] \leq \\
&\frac{1}{m+d+R}\left[dM_i + mu_i + \sum_{r=1}^{R} u_i(b^{(i),r})\right] < \\
&\frac{1}{m+d+R}\left[du_i + \frac{R\varepsilon}{2} + mu_i + R\left(u_i - \frac{\varepsilon}{2}\right)\right] \leq u_i.
\end{aligned}$$

Furthermore, since $u_i(a) \leq \nu_i(a)$ for any a, it follows from (72) that

$$\frac{1}{t}\left[\nu_i(b^{(i),t}) + \sum_{r=1}^{t-1} u_i(b^{(i),r})\right] + \frac{\varepsilon}{2} \leq u_i, \tag{75}$$

for all $1 \le i \le n$ and $1 \le t \le R$.

Next, for all $1 \le i \le n$, define the path $\pi^{(i)}$ as follows:

$$\pi^{(i),t} = \begin{cases} b^{(i),t} \text{ if } 1 \le t \le R, \\ \pi^{(0),t-R} \text{ otherwise.} \end{cases}$$

Let f be the strategy profile defined by $(\pi^{(0)}, \ldots, \pi^{(n)})$; since $\pi^{(0)}$ is confusion proof single path it follows from Proposition 1, (70) and (71) that f has 1-memory.

We now use Equations (74) and (75) to show that f is a subgame perfect equilibrium.

Consider any history $h = (s^1, \ldots, s^{t-1})$. Since $f \mid h$ induce the same outcome path as $\pi^{(0)}$ at some stage it follows that $U_i^\infty(f|h) = u_i$, for all i.

Next consider any deviation by player i to another strategy f_i' at h. Let $\pi((f_i', f_{-i}|h)) = \{\tilde{s}^\tau\}_{\tau=t}^\infty$. Since players use the limit of the mean criterion, it is enough to consider the case in which f_i' deviates from f_i infinitely often in the subgame defined by h. Thus, suppose that there is an infinite sequence $\{\mu^\tau\}_{\tau\in\mathbb{N}}$ with $\mu^\tau \ge t$, such that:

1. either $[\tilde{s}_{-i}^{\mu^\tau} = \pi_{-i}^{(0)k}$ and $\tilde{s}_i^{\mu^\tau} \neq \pi_i^{(0)k}$ for some $k \in \mathbb{N}]$ or $[\tilde{s}_{-i}^{\mu^\tau} = b_{-i}^r(i)$ and $\tilde{s}_i^{\mu^\tau} \neq b_i^r(i)]$,[11]

2. $\tilde{s}^{\mu^\tau+r} = b^r(i)$ if $1 \le r \le \min\{\mu^{\tau+1} - \mu^\tau, R+1\}$.

3. $\tilde{s}^{\mu^\tau+r} = \pi^{(0)r-R}$ if $R+1 \le r \le \mu^{\tau+1} - \mu^\tau$.

Thus,

$$\begin{aligned} U_i^\infty((f_i', f_{-i}|h)) &= \liminf_{n\to\infty} \frac{1}{n+1} \sum_{\mu=0}^{n} u_i(\tilde{s}^{t+\mu}) \\ &= \liminf_{n\to\infty} \frac{1}{\mu^{n+1} - t + 1} \left(\sum_{\mu=t}^{\mu^1} u_i(\tilde{s}^\mu) + \sum_{\tau=1}^{n} \sum_{\mu=\mu^\tau+1}^{\mu^{\tau+1}} u_i(\tilde{s}^\mu) \right) \\ &= \liminf_{n\to\infty} \frac{1}{\mu^{n+1} - \mu^1} \sum_{\tau=1}^{n} \sum_{\mu=\mu^\tau+1}^{\mu^{\tau+1}} u_i(\tilde{s}^\mu). \end{aligned}$$

Now for any τ, if $\mu^{\tau+1} - \mu^\tau = r \le R$, then

$$\sum_{\mu=\mu^\tau+1}^{\mu^{\tau+1}} u_i(\tilde{s}^\mu) \le \sum_{k=1}^{r-1} \left(u_i(b^{(i),k}) + \nu_i(b^{(i),r}) \right) \le r z_i = (\mu^{\tau+1} - \mu^\tau) u_i,$$

[11] If $\tau = 1$ it can also be that $\tilde{s}_{-i}^{\mu^1} = \pi_{-i}^{(0),k}$ and $\tilde{s}_i^{\mu^1} \neq \pi_i^{(0),k}$ for some $k \in \mathbb{N}$.

where the second inequality follows from (75).

If $\mu^{\tau+1} - \mu^{\tau} > R$, then

$$\sum_{\mu=\mu^{\tau}+1}^{\mu^{\tau+1}} u_i(\tilde{s}^{\mu}) \leq \sum_{r=1}^{R} u_i(b^{(i),r}) + \sum_{r=1}^{\mu^{\tau+1}-\mu^{\tau}-R-1} u_i(\pi^{(0),r}) + \nu_i(\pi^{(0),\mu^{\tau+1}-\mu^{\tau}-R})$$
$$< (\mu^{\tau+1} - \mu^{\tau}) u_i,$$

where the last inequality follows from (74). Therefore, $U_i^{\infty}((f_i', f_{-i}|h)) \leq u_i \leq U_i^{\infty}(f|h))$. This completes the proof of Theorem 4 ■

Proof of Theorem 3. Let $u \in \mathcal{U}$ and $\zeta > 0$. Since $\mathcal{U}^0$ is nonempty, there exists $y \in \mathcal{U}^0$ such that and $||u - y|| < \zeta/2$ (suppose $x \in \mathcal{U}^0$ and consider $y = \lambda u + (1-\lambda)x$ for some $\lambda \in (0,1)$ sufficiently close to 1).

Let $K \in \mathbb{N}$ be such that $D \subseteq \cup_{x \in D_K} B_{\zeta/4}(x)$. Then there exists a sequence $\{\bar{s}^k\}_{k=1}^{K}$ be such that

$$\left\| \frac{1}{K} \sum_{k=1}^{K} u(\bar{s}^k) - y \right\| < \frac{\zeta}{4}. \tag{76}$$

Since S_i is connected for all i, there exist a (finite) sequence $\{s^k\}_{k=1}^{K}$ such that $|u_i(s^t) - u_i(\bar{s}^t)| < \zeta/4$ for all i and t and $s_i^t \neq s_j^r$ for all $1 \leq t, r \leq K$ and $1 \leq i, j \leq n$ satisfying $(i,t) \neq (j,r)$. Then,

$$\left\| \frac{1}{K} \sum_{k=1}^{K} u(s^k) - \frac{1}{K} \sum_{k=1}^{K} u(\bar{s}^k) \right\| < \frac{\zeta}{4} \tag{77}$$

and so

$$\left\| \frac{1}{K} \sum_{k=1}^{K} u(s^k) - y \right\| < \frac{\zeta}{2}.$$

Finally, let $\tilde{\pi}$ be the repetition of $\{s^k\}_{k=1}^{K}$ and $\tilde{u} = V^{\infty}(\tilde{\pi}) = \sum_{k=1}^{K} u(s^k)/K$. Then, $\tilde{u} \in \mathcal{U}_{\mathbb{Q}}^0 \cap C$. Thus, by Theorem 4, there exists a 1 – memory subgame perfect equilibrium strategy profile f with $U^{\infty}(f) = \tilde{u}$. Moreover, since $||\tilde{u} - u|| \leq ||u - y|| + ||\tilde{u} - y|| < \zeta/2 + \zeta/2 = \zeta$, it follows that $\|U^{\infty}(f) - u\| < \zeta$. ■

Proof of Theorem 5. Let $u \in \text{int}(\mathcal{U}^0)$ and $\alpha \in \text{int}(\mathcal{U}^0)$ be such that $u_i > \alpha_i$ for all i. Since $\text{int}(\mathcal{U}^0)$ is non-empty it follows that $\dim(\mathcal{U}) = n$. Then, by Proposition 5, there exists $\hat{\delta}$ such that that for all $\delta \geq \hat{\delta}$ the following holds: for all payoffs $u \in \tilde{C}(\alpha, \delta)$ there exists 1 – memory subgame perfect equilibrium f with $U(f, \delta) = u$.

Let $\varepsilon = \min_i(u_i - v_i)$. Also, denote the discount factor corresponding to ε, given in Lemma 2 of Fudenberg and Maskin (1991), by $\bar{\delta}$.

Let $\delta^* = \max\{\hat{\delta}, \bar{\delta}\}$ and $\{s_t\}_{t=1}^{\infty}$ be a sequence of actions whose payoff is u and whose continuation payoffs at each time are within ε of u. Hence, the continuation payoffs at each time are above α. Hence, since $\delta \geq \hat{\delta}$, by the definition of $\hat{\delta}$ there exists 1 – memory subgame perfect equilibrium strategy f with $U(f, \delta) = u$. ■

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