On linear systems of conics over finite fields

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\(^1\)This research was supported by the The Scientific and Technological Research Council of Turkey, TÜBİTAK (project no. 118F159).
Overview

1. Some general context
2. Our case: conics over finite fields
3. Some history
4. Recent results
5. Remaining questions
Overview

1. Some general context
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A form on $\mathbb{P}^n$ is a homogeneous polynomial $f \in \mathbb{F}[X_0, \ldots, X_n]$, and the forms of degree $d$ on $\mathbb{P}^n$ form a vector space $W$ of dimension

$$\dim W = \binom{n + d}{d}.$$ 

A hypersurface of degree $d$ in $\mathbb{P}^n$ is the zero locus

$$\mathcal{Z}(f) = \{ p \in \mathbb{P}^n : f(p) = 0 \}$$

of a form $f$ on $\mathbb{P}^n$ of degree $d$.

A ($k$-dimensional) linear system of hypersurfaces of degree $d$ is a ($k$-dimensional) subspace of $\mathbb{P}W$. Linear systems of dimension 1, 2, and 3 are called pencils, nets, and webs, respectively.
A classification of the $k$-dimensional linear systems of hypersurfaces of degree $d$ in $\mathbb{P}^n$: a list of the (representatives of the) orbits of the $k$-dimensional subspaces of $\mathbb{P}W$ under the “induced” action of $\text{PGL}(n + 1, F)$ (the projectivity group of $\mathbb{P}^n$) on $\mathbb{P}W$.

If possible: a complete invariant for each orbit, the stabiliser group of each representative, the size of each orbit, an algorithm to determine the orbit of a given linear system, Schreier elements, ...

Two obvious questions:

1. Is such a classification possible?
2. Is the number of orbits finite?
Classification: finite number of orbits \((k = 0)\)

For hypersurfaces \((k = 0)\) we have the following theorem.

**Theorem**

Let \(W\) denote the vector space of forms of degree \(d\) on \(\mathbb{P}^n\). If \(\mathbb{F}\) is infinite, then \(\text{PGL}(n + 1, \mathbb{F})\) has a finite number of orbits on \(\mathbb{P}W\) if and only if \(d \leq 2\) or \((n, d) = (1, 3)\).

**Proof.**

(Sketch) Consider \(\text{PGL}(n + 1, \mathbb{F})\) as an algebraic group \(G\). For a hypersurface \(H\) of \(\mathbb{P}^n\) of degree \(d\) we have

\[
\dim(G) = \dim(H^G) + \dim(\text{stab}_G(H)).
\]

This implies that \(\dim(H^G) \leq n^2 + 2n\). In order to have a finite number of orbits, it is therefore required that \(\dim\mathbb{P}W \leq n^2 + 2n\). \(\square\)
Finite number of orbits \( (k = 0) \)

If \( d = 1 \) then there is one orbit of hyperplanes in \( \mathbb{P}^n \).

If \( d = 2 \) (quadrics in \( \mathbb{P}^n \)) then Sylvester’s law of inertia gives a classification for \( \mathbb{F} = \mathbb{R} \). For algebraically closed fields there is one orbit for each rank. For finite fields there are one or two orbits for each rank.

The case \( (n, d) = (1, 3) \) corresponds to cubics on \( \mathbb{P}^1 \).
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A quadratic form $f$ on $\mathbb{P}^2$ defines a conic $C = \mathcal{Z}(f)$ in $\mathbb{P}^2$.

From now on, consider the case $(n, d) = (2, 2)$, $\dim \mathbb{P}W = 5$.

Linear systems of conics correspond to subspaces of $\mathbb{P}^5$.

Over $\mathbb{F}_q$, up to $\text{PGL}(3, q)$-equivalence, there are 4 orbits of conics (classification of 0-dimensional linear systems):

- irreducible conics ($|C(\mathbb{F}_q)| = q + 1$),
- pairs of two distinct lines defined over $\mathbb{F}_q$ ($|C(\mathbb{F}_q)| = 2q + 1$),
- double lines defined over $\mathbb{F}_q$ ($|C(\mathbb{F}_q)| = q + 1$),
- pairs of $\mathbb{F}_q$-conjugate lines defined over $\mathbb{F}_{q^2}$ ($|C(\mathbb{F}_q)| = 1$).
1. Some general context
2. Our case: conics over finite fields
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History of linear systems of conics over finite fields

0-dimensional linear systems of conics: see above.

[Dickson 1907] pencils of conics over $\mathbb{F}_q$, $q$ odd (15 orbits, complete)

[Wilson 1914] nets of conics over $\mathbb{F}_q$, $q$ odd (incomplete)

[Campbell 1927] pencils of conics over $\mathbb{F}_q$, $q$ even (incomplete)

[Campbell 1928] nets of conics over $\mathbb{F}_q$, $q$ even (incomplete)

[Hirschfeld 1998] states the classification of pencils, but the result for $q$ even is attributed to [Campbell 1927], who neither stated nor proved a complete classification; he also refers to [Wilson 1914] and [Campbell 1927] for the classification of nets, but these classifications are not complete.
1. Some general context
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Linear systems of conics: recent results

(Ongoing joint work with Nour Alnajjarine, Tomasz Popiel, John Sheekey, Melissa Lee)

[ML- Popiel 2020] pencils of conics over finite fields $\mathbb{F}_q$, $q$ odd (15 orbits), and over algebraically closed fields of characteristic $\neq 2$.

[Alnajjarine - ML - Popiel 2021] pencils of conics over $\mathbb{F}_q$, $q$ even (15 orbits).

<table>
<thead>
<tr>
<th>Class/Set</th>
<th>Orbit(s)</th>
<th>Class/Set</th>
<th>Orbit(s)</th>
<th>Class/Set</th>
<th>Orbit(s)</th>
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<tbody>
<tr>
<td>Class 1</td>
<td>$\Omega_3$</td>
<td>Class 7</td>
<td>$\Omega_9$</td>
<td>Class 13</td>
<td>$\Omega_{10}$</td>
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<tr>
<td>Class 2</td>
<td>$\Omega_5$</td>
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<td>$\Omega_{12}$</td>
<td>Set 14</td>
<td>$\Omega_{14}$</td>
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<td>$\Omega_1$</td>
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<tr>
<td>Class 4</td>
<td>$\Omega_2$</td>
<td>Set 10</td>
<td>$\Omega_9, \Omega_{12, \Omega_{13}}$</td>
<td>Set 16</td>
<td>$\Omega_{15}$</td>
</tr>
<tr>
<td>Class 5</td>
<td>$\Omega_7$</td>
<td>Class 11</td>
<td>$\Omega_{11}$</td>
<td>Set 17</td>
<td>$\Omega_{15}$</td>
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<tr>
<td>Class 6</td>
<td>$\Omega_6$</td>
<td>Class 12</td>
<td>$\Omega_4$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

[ML- Popiel - Sheekey 2020] nets of conics of rank one over $\mathbb{F}_q$, $q$ odd (15 orbits).

Our approach is quite different than the approach taken by Dickson, Wilson and Campbell, which was mainly algebraic in nature, and which often relied on the computation of explicit coordinate transformations.

We use a more geometric approach by focussing on the connection between conics in $\mathbb{P}^2$ and hyperplanes in $\mathbb{P}^5$, and using the geometric and combinatorial properties of the Veronese surface in $\mathbb{P}^5$.

We also introduce some new combinatorial invariants.
The Veronese surface in $\mathbb{P}^5$

Conics of $\mathbb{P}^2$ correspond to hyperplane sections of the Veronese surface in $\mathbb{P}^5$.

We represent points $y = (y_0, y_1, y_2, y_3, y_4, y_5, y_6)$ of $\mathbb{P}^5$ by symmetric matrices

$$M_y = \begin{bmatrix}
y_0 & y_1 & y_2 \\
y_1 & y_3 & y_4 \\
y_2 & y_4 & y_5
\end{bmatrix}. \quad (1)$$

The Veronese surface $\mathcal{V}(F)$ in $\mathbb{P}^5$ is defined by setting the $2 \times 2$ minors of the above matrix to zero, and we have the corresponding Veronese map from $\mathbb{P}^2$ to $\mathcal{V}(F) \subset \mathbb{P}^5$:

$$\nu : (x_0, x_1, x_2) \mapsto (x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2).$$

The rank of a point in $y$ in $\mathbb{P}^5$ is $\text{rank}(M_y)$. 
Conics in \( \mathbb{P}^2 \) and hyperplanes in \( \mathbb{P}^5 \)

Consider a conic \( C = \mathcal{Z}(f) \) with \( f(X_0, X_1, X_2) = \sum_{i \leq j} a_{ij} X_i X_j \).

Then a point \( p(x_0, x_1, x_2) \) lies on \( C \) in \( \mathbb{P}^2 \) if and only if \( \nu(p) \) lies in the hyperplane

\[
\mathcal{Z}(a_{00} Y_0 + a_{01} Y_1 + a_{02} Y_2 + a_{11} Y_3 + a_{12} Y_4 + a_{22} Y_5)
\]

in \( \mathbb{P}^5 \).

Let \( \delta \) denote the map from the set of conics in \( \mathbb{P}^2 \) to the set of hyperplanes in \( \mathbb{P}^5 \) which takes the conic \( C = \mathcal{Z}(f) \) to

\[
\delta(C) = \mathcal{Z}(a_{00} Y_0 + a_{01} Y_1 + a_{02} Y_2 + a_{11} Y_3 + a_{12} Y_4 + a_{22} Y_5).
\]
The action of $\text{PGL}(3, q)$ on $\mathbb{P}^2$ lifts to an action on $\mathbb{P}^5$

Let $\mathcal{K} = \alpha(\text{PGL}(3, F)) \leq \text{PGL}(6, F)$ with

$$\alpha(\varphi_A) \in \text{PGL}(6, F) : y \mapsto z \text{ where } M_z = AM_yA^T.$$ 

Hyperplanes of $\mathbb{P}^5$: 4 $\mathcal{K}$-orbits

- $\mathcal{H}_1$ is the $\mathcal{K}$-orbit of hyperplanes intersecting $\mathcal{V}(F)$ in a conic,
- $\mathcal{H}_3$ is the $\mathcal{K}$-orbit of hyperplanes intersecting $\mathcal{V}(F)$ in a normal rational curve,
- $\mathcal{H}_2$ is the set of hyperplanes not contained in $\mathcal{H}_1 \cup \mathcal{H}_3$.

The set $\mathcal{H}_2$ splits into to further $\mathcal{K}$-orbits.
Extend the definition of $\delta$ from the set of conics to the set of linear systems of conics as follows. Given any set $S$ of conics in $\mathbb{P}^2$, define

$$\delta(S) = \cap_{C \in S} \delta(C).$$

In this way we obtain the following one-to-one correspondences.

**Lemma**

*The equivalence classes of pencils, resp. nets, of conics in $\mathbb{P}^2$ are in one-to-one correspondence with the $K$-orbits of solids, resp. planes, in $\mathbb{P}^5$.***
The pencils of conics correspond to solids in $\mathbb{P}^5$.

If $q$ is odd, then the classification of the $K$-orbits of solids follows from the classification of $K$-orbits of lines in $\mathbb{P}^5$ by virtue of a suitable polarity of $\mathbb{P}^5$. The classification therefore follows from [ML-Popiel 2020]. (Confirming the 15 equivalent classes of pencils of conics obtained by Dickson.)

For $q$ is even, the classification of solids was recently completed in [Alnajjarine-ML-Popiel 2021]. (Completing and correcting Campbell’s results.)

(Note: these papers also contain the stabiliser groups, and other combinatorial invariants of the $K$-orbits.)
Nets of conics

A net of conics $\mathcal{N}$ is defined by three conics $C_i = \mathcal{Z}(f_i)$ ($i = 1, 2, 3$), not contained in a pencil:

$$\mathcal{N} = \{ \mathcal{Z}(af_1 + bf_2 + cf_3) : a, b, c \in \mathbb{F}, (a, b, c) \neq (0, 0, 0) \}.$$ 

Then

$$xf_1 + yf_2 + zf_3 = a_{00}(x, y, z)X_0^2 + a_{01}(x, y, z)X_0X_1 + \cdots + a_{22}(x, y, z)X_2^2$$

is a quadratic form whose coefficients are linear forms in $x, y, z$.

We thus obtain a symmetric matrix $A_{\mathcal{N}}$ over $\mathbb{F}[x, y, z]$. 
Nets of conics of rank one

\((\text{char} \mathbb{F} \neq 2)\)

We define the discriminant of the net \(\mathcal{N}\) as

\[ \Delta_{\mathcal{N}} = \det(A_{\mathcal{N}}). \]

The discriminant \(\Delta_{\mathcal{N}}\) defines a cubic curve \(Z(\Delta_{\mathcal{N}})\) in \(\mathbb{P}^2\).

For each \(a, b, c \in \mathbb{F}_q\), not all zero, we obtain a conic

\[ \mathcal{N}(a, b, c) = Z(af_1 + bf_2 + cf_3). \]

Lemma

The conic \(\mathcal{N}(a, b, c)\) is singular if and only if \((a, b, c)\) lies on the cubic \(Z(\Delta_{\mathcal{N}})\).

A net has rank one if it contains a repeated line.

(rank two if it contains no repeated lines but contains a conic which is not absolutely irreducible, and rank three if every conic in the net is absolutely irreducible)
$K$-invariants

From now on assume that $\mathbb{F} = \mathbb{F}_q$ with $q$ odd.

There are four $K$-orbits of points in $\mathbb{P}^5$, namely: $\mathcal{P}_1$, the points of rank 1; $\mathcal{P}_{2,e}$, the exterior rank-2 points; $\mathcal{P}_{2,i}$, the interior rank-2 points; and $\mathcal{P}_3$, the points of rank 3.

The point-orbit distribution of a subspace $W$ is the list $[n_1, n_2, n_3, n_4]$, where $n_i$ is the number of points in $W$ which belong to the $i$-th point-orbit.

The rank distribution of $W$ is $[n_1, n_2 + n_3, n_4]$.

The line-orbits were determined in [ML-Popiel 2020] using [ML-Sheekey]'s classification of tensors in $\mathbb{F}_q^2 \otimes \mathbb{F}_q^3 \otimes \mathbb{F}_q^3$.

Their point-orbit distributions and stabilisers were determined in [ML-Popiel 2020].
Representatives of the $K$-orbits on lines in $\mathbb{P}^5$

<table>
<thead>
<tr>
<th>Orbit</th>
<th>Representative</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$o_5$</td>
<td>$\begin{bmatrix} \alpha &amp; \cdot &amp; \cdot \ \cdot &amp; \beta &amp; \cdot \ \cdot &amp; \cdot &amp; \cdot \end{bmatrix}$</td>
<td>$\alpha \beta$</td>
</tr>
<tr>
<td>$o_6$</td>
<td>$\begin{bmatrix} \alpha &amp; \beta &amp; \cdot \ \cdot &amp; \cdot &amp; \cdot \ \cdot &amp; \cdot &amp; \cdot \end{bmatrix}$</td>
<td>$\alpha \beta$</td>
</tr>
<tr>
<td>$o_{8,1}$</td>
<td>$\begin{bmatrix} \alpha &amp; \cdot &amp; \cdot \ \cdot &amp; \beta &amp; \cdot \ \cdot &amp; \cdot &amp; \cdot \end{bmatrix}$</td>
<td>$\beta \in \mathbb{F}_q$</td>
</tr>
<tr>
<td>$o_{8,2}$</td>
<td>$\begin{bmatrix} \alpha &amp; \cdot &amp; \cdot \ \cdot &amp; \beta &amp; \cdot \ \cdot &amp; \cdot &amp; \cdot \end{bmatrix}$</td>
<td>$\beta \in \mathbb{F}_q$</td>
</tr>
<tr>
<td>$o_9$</td>
<td>$\begin{bmatrix} \alpha &amp; \beta \ \beta &amp; \cdot \ \cdot &amp; \cdot \end{bmatrix}$</td>
<td>$\beta \cdot u\alpha + \beta$</td>
</tr>
<tr>
<td>$o_{10}$</td>
<td>$\begin{bmatrix} \beta &amp; \alpha + u\beta \ \cdot &amp; \cdot \end{bmatrix}$</td>
<td>$\beta \cdot u\alpha + \beta$</td>
</tr>
<tr>
<td>$o_{12}$</td>
<td>$\begin{bmatrix} \alpha &amp; \beta \ \cdot &amp; \cdot \end{bmatrix}$</td>
<td>$\beta \cdot u\beta - w\alpha \cdot \alpha$</td>
</tr>
<tr>
<td>$o_{13,1}$</td>
<td>$\begin{bmatrix} \alpha &amp; \cdot &amp; \cdot \ \cdot &amp; \beta &amp; \cdot \ \cdot &amp; \cdot &amp; \cdot \end{bmatrix}$</td>
<td>$\epsilon \in \mathbb{F}_q$</td>
</tr>
<tr>
<td>$o_{13,2}$</td>
<td>$\begin{bmatrix} \alpha &amp; \cdot &amp; \cdot \ \cdot &amp; \beta &amp; \cdot \ \cdot &amp; \cdot &amp; \cdot \end{bmatrix}$</td>
<td>$\epsilon \in \mathbb{F}_q$</td>
</tr>
<tr>
<td>$o_{14,1}$</td>
<td>$\begin{bmatrix} \alpha &amp; \cdot &amp; \cdot \ \cdot &amp; -(\alpha + \beta) &amp; \cdot \ \cdot &amp; \cdot &amp; \cdot \end{bmatrix}$</td>
<td>$\epsilon \in \mathbb{F}_q$</td>
</tr>
<tr>
<td>$o_{14,2}$</td>
<td>$\begin{bmatrix} \alpha &amp; \cdot &amp; \cdot \ \cdot &amp; -\epsilon(\alpha + \beta) &amp; \cdot \ \cdot &amp; \cdot &amp; \cdot \end{bmatrix}$</td>
<td>$\epsilon \in \mathbb{F}_q$</td>
</tr>
<tr>
<td>$o_{15,1}$</td>
<td>$\begin{bmatrix} v\beta &amp; \alpha \ \cdot &amp; \cdot \end{bmatrix}$</td>
<td>$-v \in \mathbb{F}_q, (*)$</td>
</tr>
<tr>
<td>$o_{15,2}$</td>
<td>$\begin{bmatrix} v\beta &amp; \alpha \ \cdot &amp; \cdot \end{bmatrix}$</td>
<td>$-v \in \mathbb{F}_q, (*)$</td>
</tr>
<tr>
<td>$o_{16}$</td>
<td>$\begin{bmatrix} \alpha &amp; \cdot \ \cdot &amp; \beta \cdot \end{bmatrix}$</td>
<td>$\alpha \beta$</td>
</tr>
<tr>
<td>$o_{17}$</td>
<td>$\begin{bmatrix} v^{-1}\alpha &amp; \beta \ \beta &amp; u\beta - w\alpha \alpha \ \cdot &amp; \cdot \end{bmatrix}$</td>
<td>$\alpha \beta$</td>
</tr>
</tbody>
</table>

Conditions (*) and (**) hold for all $\lambda \in \mathbb{F}_q$: $\lambda^2 + u\alpha \lambda - 1 \neq 0$, and $\lambda^3 + w\lambda^2 - u\lambda + v \neq 0$ for all $\lambda \in \mathbb{F}_q$. 


Point-orbit distributions of the $K$-orbits on lines in $\mathbb{P}^5$

<table>
<thead>
<tr>
<th>Orbit</th>
<th>Point-orbit distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$o_5$</td>
<td>$[2, \frac{q-1}{2}, \frac{q-1}{2}, 0]$</td>
</tr>
<tr>
<td>$o_6$</td>
<td>$[1, q, 0, 0]$</td>
</tr>
<tr>
<td>$o_{8,1}$</td>
<td>$[1, 1, 0, q - 1]$</td>
</tr>
<tr>
<td>$o_{8,2}$</td>
<td>$[1, 0, 1, q - 1]$</td>
</tr>
<tr>
<td>$o_9$</td>
<td>$[1, 0, 0, q]$</td>
</tr>
<tr>
<td>$o_{10}$</td>
<td>$[0, \frac{q+1}{2}, \frac{q+1}{2}, 0]$</td>
</tr>
<tr>
<td>$o_{12}$</td>
<td>$[0, q + 1, 0, 0]$</td>
</tr>
<tr>
<td>$o_{13,1}$</td>
<td>$[0, 2, 0, q - 1]$</td>
</tr>
<tr>
<td>$o_{13,2}$</td>
<td>$[0, 1, 1, q - 1]$</td>
</tr>
<tr>
<td>$o_{14,1}$</td>
<td>$[0, 3, 0, q - 2]$</td>
</tr>
<tr>
<td>$o_{14,2}$</td>
<td>$[0, 1, 2, q - 2]$</td>
</tr>
<tr>
<td>$o_{15,1}$</td>
<td>$[0, 1, 0, q]$</td>
</tr>
<tr>
<td>$o_{15,2}$</td>
<td>$[0, 0, 1, q]$</td>
</tr>
<tr>
<td>$o_{16}$</td>
<td>$[0, 1, 0, q]$</td>
</tr>
<tr>
<td>$o_{17}$</td>
<td>$[0, 0, 0, q + 1]$</td>
</tr>
</tbody>
</table>
Steps in the classification

1. Planes spanned by points of $\mathcal{V}(\mathbb{F}_q)$: $\Sigma_1$ and $\Sigma_2$.
2. Planes meeting the quadric Veronesean in exactly two points: $\Sigma_3$, $\Sigma_4$ and $\Sigma_5$.
3. Planes that are spanned by points of rank $\leq 2$ and meet the quadric Veronesean in exactly one point: eight further orbits $\Sigma_6, \ldots, \Sigma_{13}$ (independent of $q$), one orbit $\Sigma_{14}$ which only appears in characteristic $\neq 3$, and one orbit $\Sigma'_{14}$ which only appears in characteristic 3.
4. Exactly one remaining orbit, $\Sigma_{15}$, consisting of planes that meet the Veronesean but are not spanned by points in the secant variety of the Veronesean.
Classification

**Theorem (ML - Popiel - Sheekey 2020)**

Let $q$ be a power of an odd prime. There are 15 orbits of planes in $\mathbb{P}^5$ that meet the quadric Veronesean in at least one point, under the action of $\text{PGL}(3, q) \leq \text{PGL}(6, q)$.

As a corollary, we complete Wilson’s classification of nets of rank one, rectifying some of the statements made in his paper.

**Corollary**

There are 15 orbits of nets of conics of rank one in $\mathbb{P}^2(\mathbb{F}_q)$, $q$ odd.
Wilson was aware of the fact that he had not completely classified the orbits, pointing out that

"All questions of inter-relations between these types have been considered and answered, except ..."

Although Wilson’s work was in general very thorough, there are also some other issues with his classification besides the open cases he mentioned; there are some inaccuracies, and some missing orbits.

<table>
<thead>
<tr>
<th>$\Sigma_1$</th>
<th>$\Sigma_2$</th>
<th>$\Sigma_3$</th>
<th>$\Sigma_4$</th>
<th>$\Sigma_5$</th>
<th>$\Sigma_6$</th>
<th>$\Sigma_7$</th>
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<th>$\Sigma_9$</th>
<th>$\Sigma_{10}$</th>
<th>$\Sigma_{11}$</th>
<th>$\Sigma_{12}$</th>
<th>$\Sigma_{13}$</th>
<th>$\Sigma_{14}$</th>
<th>$\Sigma'_{14}$</th>
<th>$\Sigma_{15}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>III</td>
<td>IV</td>
<td>VI</td>
<td>XI</td>
<td>VIII</td>
<td>I</td>
<td>V</td>
<td>VII</td>
<td>VII</td>
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<td>XI</td>
<td>X, XI</td>
<td>X, XI</td>
<td>II</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1. Correspondence between $K$-orbits of planes in $\mathbb{P}^5$ and canonical types $I, \ldots, XI$ of nets of conics of rank one in $\mathbb{P}^2$ obtained by [Wilson 1941].
Representatives of the $K$-orbits on planes in $\mathbb{P}^5$

<table>
<thead>
<tr>
<th>Orbit</th>
<th>Representative</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Sigma_1$</td>
<td>$\begin{bmatrix} \alpha &amp; \gamma \ \gamma &amp; \beta \ \cdot &amp; \cdot \end{bmatrix}$</td>
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</tr>
<tr>
<td>$\Sigma_2$</td>
<td>$\begin{bmatrix} \cdot &amp; \beta \ \cdot &amp; \cdot \ \alpha &amp; \cdot \end{bmatrix}$</td>
<td></td>
</tr>
<tr>
<td>$\Sigma_3$</td>
<td>$\begin{bmatrix} \alpha &amp; \cdot \ \cdot &amp; \cdot \ \gamma &amp; \cdot \end{bmatrix}$</td>
<td></td>
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<tr>
<td>$\Sigma_4$</td>
<td>$\begin{bmatrix} \cdot &amp; \cdot \ \gamma &amp; \cdot \ \cdot &amp; \cdot \end{bmatrix}$</td>
<td></td>
</tr>
<tr>
<td>$\Sigma_5$</td>
<td>$\begin{bmatrix} \gamma &amp; \cdot \ \cdot &amp; \cdot \ \alpha &amp; \beta \end{bmatrix}$</td>
<td></td>
</tr>
<tr>
<td>$\Sigma_6$</td>
<td>$\begin{bmatrix} \cdot &amp; \cdot \ \cdot &amp; \cdot \ \alpha &amp; \beta \end{bmatrix}$</td>
<td>$\varepsilon \in \mathbb{C}$</td>
</tr>
<tr>
<td>$\Sigma_7$</td>
<td>$\begin{bmatrix} \cdot &amp; \cdot \ \gamma &amp; \cdot \ \cdot &amp; \cdot \end{bmatrix}$</td>
<td></td>
</tr>
<tr>
<td>$\Sigma_8$</td>
<td>$\begin{bmatrix} \beta &amp; \cdot \ \cdot &amp; \gamma \end{bmatrix}$</td>
<td></td>
</tr>
<tr>
<td>$\Sigma_9$</td>
<td>$\begin{bmatrix} \alpha &amp; \cdot \ \cdot &amp; \cdot \ \cdot &amp; -\gamma \end{bmatrix}$</td>
<td></td>
</tr>
<tr>
<td>$\Sigma_{10}$</td>
<td>$\begin{bmatrix} \cdot &amp; \beta \ \cdot &amp; \cdot \ \beta &amp; \cdot \end{bmatrix}$</td>
<td>$\varepsilon \in \mathbb{C}$</td>
</tr>
<tr>
<td>$\Sigma_{11}$</td>
<td>$\begin{bmatrix} \beta &amp; \cdot \ \cdot &amp; \cdot \ \cdot &amp; -\varepsilon \cdot \gamma \end{bmatrix}$</td>
<td></td>
</tr>
<tr>
<td>$\Sigma_{12}$</td>
<td>$\begin{bmatrix} \beta &amp; \cdot \ \cdot &amp; \cdot \ \cdot &amp; \cdot \end{bmatrix}$</td>
<td></td>
</tr>
<tr>
<td>$\Sigma_{13}$</td>
<td>$\begin{bmatrix} \beta &amp; \cdot \ \cdot &amp; \cdot \ \cdot &amp; \varepsilon \cdot \gamma \end{bmatrix}$</td>
<td>$\varepsilon \in \mathbb{C}$</td>
</tr>
<tr>
<td>$\Sigma_{14}$</td>
<td>$\begin{bmatrix} \alpha &amp; \beta \ \beta &amp; \cdot \gamma \ \cdot &amp; \cdot \gamma \end{bmatrix}$</td>
<td>$q \neq 0 \pmod{3}$, $(\dagger)$</td>
</tr>
<tr>
<td>$\Sigma_{14}'$</td>
<td>$\begin{bmatrix} \cdot &amp; \cdot \ \alpha + \gamma &amp; \cdot \ \gamma &amp; \cdot \end{bmatrix}$</td>
<td>$q \equiv 0 \pmod{3}$</td>
</tr>
<tr>
<td>$\Sigma_{15}$</td>
<td>$\begin{bmatrix} \alpha &amp; \cdot \ \cdot &amp; \gamma \end{bmatrix}$</td>
<td></td>
</tr>
</tbody>
</table>
Point-orbit distributions of the $K$-orbits on planes in $\mathbb{P}^5$

<table>
<thead>
<tr>
<th>Orbit</th>
<th>Point-orbit distribution</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Sigma_1$</td>
<td>$[q + 1, q(q + 1)/2, q(q - 1)/2, 0]$</td>
<td>$q \equiv \pm 1 \pmod{3}$</td>
</tr>
<tr>
<td>$\Sigma_2$</td>
<td>$[3, 3(q - 1)/2, 3(q - 1)/2, q^2 - 2q + 1]$</td>
<td>$q \equiv 0 \pmod{3}$</td>
</tr>
<tr>
<td>$\Sigma_3$</td>
<td>$[2, (3q - 1)/2, (q - 1)/2, q^2 - q]$</td>
<td></td>
</tr>
<tr>
<td>$\Sigma_4$</td>
<td>$[2, (3q - 1)/2, (q - 1)/2, q^2 - q]$</td>
<td></td>
</tr>
<tr>
<td>$\Sigma_5$</td>
<td>$[2, q - 1, q - 1, q^2 - q + 1]$</td>
<td></td>
</tr>
<tr>
<td>$\Sigma_6$</td>
<td>$[1, (q + 1)/2, (q + 1)/2, q^2 - 1]$</td>
<td></td>
</tr>
<tr>
<td>$\Sigma_7$</td>
<td>$[1, q^2 + q, 0, 0]$</td>
<td></td>
</tr>
<tr>
<td>$\Sigma_8$</td>
<td>$[1, 2q, 0, q^2 - q]$</td>
<td></td>
</tr>
<tr>
<td>$\Sigma_9$</td>
<td>$[1, 2q, 0, q^2 - q]$</td>
<td></td>
</tr>
<tr>
<td>$\Sigma_{10}$</td>
<td>$[1, q, q^2 - q]$</td>
<td></td>
</tr>
<tr>
<td>$\Sigma_{11}$</td>
<td>$[1, q, 0, q^2]$</td>
<td></td>
</tr>
<tr>
<td>$\Sigma_{12}$</td>
<td>$[1, (q - 1)/2, (q - 1)/2, q^2 + 1]$</td>
<td></td>
</tr>
<tr>
<td>$\Sigma_{13}$</td>
<td>$[1, (q + 1)/2, (q + 1)/2, q^2 - 1]$</td>
<td></td>
</tr>
<tr>
<td>$\Sigma_{14}$</td>
<td>$[1, (q \mp 1)/2, (q \mp 1)/2, q^2 \pm 1]$</td>
<td>$q \equiv \pm 1 \pmod{3}$</td>
</tr>
<tr>
<td>$\Sigma'_{14}$</td>
<td>$[1, q, 0, q^2]$</td>
<td>$q \equiv 0 \pmod{3}$</td>
</tr>
<tr>
<td>$\Sigma_{15}$</td>
<td>$[1, q, 0, q^2]$</td>
<td></td>
</tr>
</tbody>
</table>
3. Planes spanned by $\mathcal{V}(\mathbb{F}_q)^{(2)}$ and meeting $\mathcal{V}(\mathbb{F}_q)$ in one point.

(→ orbits $\Sigma_6, \ldots, \Sigma_{13}$, and $\Sigma_{14}$ or $\Sigma_1'$ depending on $q \mod 3$)

**Lemma**

A point $y$ of rank two defines a unique conic $C_y$ lying on $\mathcal{V}(\mathbb{F}_q)$.

Let $\pi = \langle x, y, z \rangle$, with $x \in \mathcal{V}(\mathbb{F}_q)$ and $y, z \in \mathcal{V}(\mathbb{F}_q)^{(2)} \setminus \mathcal{V}(\mathbb{F}_q)$.

Consider the conics $C_y$ and $C_z$ determined by $y$ and $z$, and let $p_x$ and $\ell_y, \ell_z$ be such that $x = \nu(p_x)$, $C_y = \nu(\ell_y)$ and $C_z = \nu(\ell_z)$.

If $x \in C_y = C_z$, then $\pi$ is a conic plane, and so lies in the orbit $\Sigma_1$. 
The remaining possibilities (up to symmetry) are as follows:

(a) \( x \notin C_y = C_z, \ (\rightarrow \Sigma_6) \)
(b) \( x = C_y \cap C_z, \ (\rightarrow \Sigma_7) \)
(c) \( x \in C_y \setminus C_z, \ (\rightarrow \Sigma_8, \Sigma_9, \Sigma_{10}) \)
(d) \( C_y \neq C_z \) and \( x \notin C_y \cup C_z \).
(d) Suppose that $x \notin C_y \cup C_z$, and write $w = C_y \cap C_z$.

Without loss of generality, we may fix $p_w$, $p_x$, $\ell_y$ and $\ell_z$.

Then the stabiliser of this configuration in $\mathbb{P}^2$ is contained in the group of perspectivities with centre $p_w$.

On the plane $\langle C_y \rangle$ in $\mathbb{P}^5$, the induced group acts as the stabiliser of the conic $C_y$ and the point $w$. There are two possibilities:

(i) $y$ lies on the tangent line $t_w(C_y)$,

(ii) $y$ does not lie on $t_w(C_y)$. 
(d-i) Suppose that $y$ lies on the tangent line $t_w(C_y)$. Let $u \in \mathbb{P}^5$ be such that $y = t_u(C_y) \cap t_w(C_y)$. Then the stabiliser of $x$, $y$, $w$ and $C_z$ is induced by the group of perspectivities with centre $p_w$ and axis $\langle p_x, p_u \rangle$, and acts on $\langle C_z \rangle$ as the stabiliser of $C_z$ and the two points $w$ and $v := \nu(\langle p_x, p_u \rangle \cap \ell_z)$ of $C_z$. Consider $t_v(C_z)$ to $C_z$.

There are five possibilities for $z$:

(A) $z = t_w(C_y) \cap t_v(C_z)$,

(B) $z \in t_w(C_y) \setminus t_v(C_z)$,

(C) $z \in t_v(C_z) \setminus t_w(C_y)$,

(D) $z \in \langle w, v \rangle$, and

(E) $z \notin t_v(C_z) \cup t_w(C_y) \cup \langle w, v \rangle$. 
(d-i-D) For convenience, fix \( p_w, p_x, p_u \) and \( p_v \) as \( \langle e_2 \rangle, \langle e_1 \rangle, \langle e_1 + e_3 \rangle \) and \( \langle e_3 \rangle \), respectively. This implies that \( \ell_y = \langle e_1 + e_3, e_2 \rangle \) and \( \ell_z = \langle e_2, e_3 \rangle \). Note that \( y \) is now the point with coordinates \((0, 1, 0, 0, 1, 0)\). Since we are now assuming that \( z \) is on the line \( \langle w, v \rangle \), we may take \( z = (0, 0, 0, 1, 0, b) \) for some non-zero \( b \in \mathbb{F}_q \).

Then we obtain the orbits represented by

\[
\Sigma_{12} : \begin{bmatrix} \alpha & \beta & \cdot \\ \beta & \gamma & \beta \\ \cdot & \beta & \gamma \end{bmatrix} \quad \text{and} \quad \Sigma_{13} : \begin{bmatrix} \alpha & \beta & \cdot \\ \beta & \gamma & \beta \\ \cdot & \beta & \varepsilon \gamma \end{bmatrix},
\]

where \( \varepsilon \) is a non-square in \( \mathbb{F}_q \) \((b = 1, b = \varepsilon)\).
The point-orbit distributions of $\Sigma_{12}$ and $\Sigma_{13}$

Lemma (implies $\Sigma_{12}$ and $\Sigma_{13}$ are $\neq K$-orbits)

A plane belonging to the $K$-orbit $\Sigma_{12}$ has point-orbit distribution

$$[1, (q - 1)/2, (q - 1)/2, q^2 + 1],$$

and a plane belonging to the $K$-orbit $\Sigma_{13}$ has point-orbit distribution

$$[1, (q + 1)/2, (q + 1)/2, q^2 - 1].$$

Let $\pi_{12}, \pi_{13}$ denote these representatives, and let $C_{12}, C_{13}$ denote the respective cubics. Given a fixed basis for $\pi_i$, $i \in \{12, 13\}$, define a map from $\pi_i$ to $\mathbb{P}^2$ in the natural way, and denote the image of $C_i$ in $\mathbb{P}^2$ by $\overline{C}_i$. 
Inflexion points of the cubics $\overline{C}_{12}$ and $\overline{C}_{13}$

Lemma (is used for future $K$-orbits)

Suppose that $q$ is not a power of 3. Then the cubic curves $\overline{C}_{12}$ and $\overline{C}_{13}$ each have a double point at $P = (1, 0, 0)$. The tangents at the double point, and the points and lines of inflexion, are as given in the following table:

<table>
<thead>
<tr>
<th>Plane</th>
<th>$\pi_{12}$</th>
<th>$\pi_{13}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tangents at $P$</td>
<td>$\beta = \pm \gamma$</td>
<td>$\beta = \pm \sqrt{\epsilon \gamma}$</td>
</tr>
<tr>
<td>Inflexion points</td>
<td>${(0, 1, 0), (-1, \pm 4\sqrt{-1/3}, 4)}$</td>
<td>${(0, 1, 0), (-\epsilon, \pm 4\sqrt{-\epsilon/3}, 4)}$</td>
</tr>
<tr>
<td>Line of inflexion</td>
<td>$4\alpha + \gamma = 0$</td>
<td>$4\alpha + \epsilon \gamma = 0$</td>
</tr>
</tbody>
</table>

In particular, when $-3$ is a square in $\mathbb{F}_q$, $\overline{C}_{12}$ has three $\mathbb{F}_q$-rational inflexion points, and $\overline{C}_{13}$ has one. When $-3$ is a non-square, $\overline{C}_{12}$ has one $\mathbb{F}_q$-rational inflexion point, and $\overline{C}_{13}$ has three.
The representative $\pi_c$

(d-i-E) If the point $z$ is not on the line $\langle w, v \rangle$ and not on the tangents $t_w(C_y)$ and $t_v(C_z)$, then we may assume the line $\langle z, w \rangle$ passes through a point $r \in C_z \setminus \{w, v\}$. We retain the same representatives as in case (d-i-D), and without loss of generality we further choose $r = \nu(p_r)$ with $p_r = \langle e_2 - e_3 \rangle$. Then we may take $z = z_c := (0, 0, 0, c, -1, 1)$ where $c \not\in \{0, 1\}$ (because $z_0$ lies on $t_v(C_z)$ and $z_1 = r$). The plane $\pi_c = \langle x, y, z_c \rangle$ is then represented by the matrix

$$
\begin{bmatrix}
\alpha & \beta & \cdot \\
\beta & c\gamma & \beta - \gamma \\
\cdot & \beta - \gamma & \gamma
\end{bmatrix}.
$$
Does $\pi_c$ give a new orbit (new orbits)?

$$(c \in \mathbb{F}_q \setminus \{0, 1\})$$

**Lemma**
If $-3c \neq 0$ is a square in $\mathbb{F}_q$ and $\frac{\sqrt{c+1}}{\sqrt{c-1}}$ is not a cube in $\mathbb{F}_q(\sqrt{-3})$, then the plane $\pi_c$ is not in any of the $K$-orbits $\Sigma_1, \ldots, \Sigma_{13}$.

**Lemma**
If $\pi_c$ is contained in one of the $K$-orbits $\Sigma_1, \ldots, \Sigma_{13}$, then $\pi_c \in \Sigma_{12}$ if $c$ is a square in $\mathbb{F}_q$, and $\pi_c \in \Sigma_{13}$ otherwise.

**Lemma**
If $q$ is a power of 3 then $\pi_c \in \Sigma_{12} \cup \Sigma_{13}$. 
How many $K$-orbits do $\pi_c \notin \Sigma_i$, $i \leq 13$ give?

Lemma (at least one new $K$-orbit)

If $q$ is not a power of 3 then there exists $c \in \mathbb{F}_q$ such that the plane $\pi_c$ is not in any of the $K$-orbits $\Sigma_1, \ldots, \Sigma_{13}$.

Lemma (when is $\pi_c \sim_K \pi_d$)

The planes $\pi_c$ and $\pi_d$ belong to the same $K$-orbit if one of the following conditions holds:

(i) $cd = 1$,

(ii) $\left(\frac{\sqrt{c}-1}{\sqrt{c}+1}\right)\left(\frac{\sqrt{d}-1}{\sqrt{d}+1}\right)$ is a cube in $\mathbb{F}_q(\sqrt{-3})$,

(iii) $\left(\frac{\sqrt{c}+1}{\sqrt{c}-1}\right)\left(\frac{\sqrt{d}-1}{\sqrt{d}+1}\right)$ is a cube in $\mathbb{F}_q(\sqrt{-3})$.

Lemma (exactly one new $K$-orbit)

All "new" planes $\pi_c$ belong to the same $K$-orbit.

Conclusion. One more orbit $\Sigma_{14}$ for $q \not\equiv 0(3)$. 
The orbit $\Sigma'_14$ for $q \equiv 0(3)$

(d-ii) The only planes $\pi = \langle x, y, z \rangle$ with $\text{rank}(x) = 1$ and $\text{rank}(y) = \text{rank}(z) = 2$ that we have not yet considered are those such that $y$ is not on the tangent $t_w(C_y)$ to the conic $C_y$ through the point $w = C_y \cap C_z$ in $\langle C_y \rangle$ (and likewise $z$ is not on $t_w(C_z)$).

Lemma (no new $K$-orbit except for $q \equiv 0(3)$)

Such a plane $\pi$ belongs to a previously considered $K$-orbits except when $q \equiv 0 \pmod{3}$, in which case all those planes form a single $K$-orbit $\Sigma'_14$.

$\Sigma'_14 : \begin{bmatrix} \alpha + \gamma & \gamma & \gamma \\ \gamma & \beta + \gamma & \gamma \\ \gamma & \gamma & -\beta \end{bmatrix}$, for $q \equiv 0 \pmod{3}$. 
The final orbit $\Sigma_{15}$

Finally, planes meeting $\mathcal{V}(\mathbb{F}_q)$ in 1 pt but not spanned by $\mathcal{V}(\mathbb{F}_q)^{(2)}$.

At most one of the lines through the point $x \in \pi \cap \mathcal{V}(\mathbb{F}_q)$ meets $\mathcal{V}(\mathbb{F}_q)^{(2)} \setminus \{x\}$.

[ML-Popiel 2020] $\Rightarrow$ each such line has type $o_9$.

After applying some transformations and applying transitivity properties we can obtain

$$\Sigma_{15} : \begin{bmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \cdot \\ \gamma & \cdot & \cdot \end{bmatrix}.$$ 

Lemma

*The $K$-orbit $\Sigma_{15}$ is distinct from $\Sigma_1 \cup \cdots \cup \Sigma_{14}$, and from $\Sigma'_{14}$ when $q \equiv 0(3)$.*

This concludes the classification of nets of rank one, for $q$ odd.
Final comments

- The study of linear systems over finite fields was (almost) abandoned for almost a century (there are some (incomplete) classification results which were not mentioned here).
- In fact, until recently, the only complete such classification was the one obtained by Dickson in 1906: pencils of conics over \( \mathbb{F}_q \), \( q \) odd.
- The geometric/combinatorial approach presented here allowed for the classification of pencils (all \( q \)), nets of rank one (\( q \) odd) and webs (all \( q \)) of conics in \( \mathbb{P}^2(\mathbb{F}_q) \).
- Many classification problems are still open, but seem hard.
Results on other cases

For \((n, d) = (2, 3), k = 0\) [Schoof 1986] determined the number of projectively equivalent nonsingular cubics over \(\mathbb{F}_q\) with a fixed number of \(\mathbb{F}_q\)-rational points. ([Bedocchi 1981], [Cicchese 1965, 1971], [De Groote - Hirschfeld 1980], [Duering 1941], [Waterhouse 1969])

For \(d = 2, k = 1, \text{ and } \mathbb{F} = \mathbb{C}\): two "non-singular" pencils \(\mathcal{P}_1\) and \(\mathcal{P}_2\) of quadrics, are projectively equivalent if and only if they have the same Segre symbol and there is a projectivity on \(\mathbb{P}^1\) taking the points \((x_0^{(i)}, x_1^{(i)})\) parametrizing the singular elements of \(\mathcal{P}_1\) to those of \(\mathcal{P}_2\). [Kronecker, Weierstrass, C. Segre]

For \((n, d) = (3, 2), k = 1, \text{ and } \mathbb{F} = \mathbb{F}_q\): [Bruen - Hirschfeld 1986] [Bruen - Hirschfeld 1988] non-singular pencils of quadrics with reducible base.

For \((n, d) = (1, 3), k = 1, \text{ and } \mathbb{F} = \mathbb{F}_q, q \text{ odd, and } q \not\equiv 0(3): partial results [Blokhuis, Pellikaan, Szőnyi 2021], [Davydov, Marcugini, Pambianco 2021], [ML - Günay 2021].