Good Eggs and Veronese Varieties

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Abstract

We give a new proof of the main theorem of [6] concerning the connection between good eggs in \(\text{PG}(4n-1, q)\), \(q\) odd, and Veronese varieties, using the model for good eggs in \(\text{PG}(4n-1, q)\), \(q\) odd, from [2].

Key words: Desarguesian spreads, subgeometries, eggs

1 Introduction

An egg \(E\) in \(\text{PG}(4n-1, q)\) is a partial \((n-1)\)-spread of size \(q^{2n} + 1\), such that every three egg elements span a \((3n-1)\)-space and for every egg element \(E\) there exists a \((3n-1)\)-space \(T_E\) (called the tangent space of \(E\) at \(E\)) which contains \(E\) and is skew to the other egg elements. The egg is good at an element \(E\) if every \((3n-1)\)-space which contains \(E\) and two other egg elements, contains exactly \(q^n + 1\) egg elements. Put \(F = \text{GF}(q^n)\), \(q\) odd, and let \(E\) be a good egg of \(\text{PG}(4n-1, q)\). In [2] it was shown that there exist \(a_i, b_i, c_i \in F\), for \(i \in \{0, \ldots, n-1\}\), such that the elements of \(E\) can be written as

\[
E(a, b) = \{\langle-g_t(a, b), t, -at, -bt\rangle \| t \in F^*\}, \quad \forall a, b \in F,
\]

\[
E(\infty) = \{\langle t, 0, 0, 0\rangle \| t \in F^*\},
\]

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with
\[ g_t(a, b) = \sum_{i=0}^{n-1} (a_i a^2 + b_i ab + c_i b^2)^{1/q} t^{1/q}, \]
and with this notation the egg is good at the element \( E(\infty) \). We use the notation \( \langle x_1, \ldots, x_d \rangle \) for the projective point corresponding to the vector \((x_1, \ldots, x_d)\) and the egg elements are represented as subsets of \( \text{PG}(3, q^n) \). Starting from an ovoid of \( \text{PG}(3, q^n) \) one can construct an egg of \( \text{PG}(4n-1, q) \), and we call such an egg elementary ([5]).

2 A theorem by J. A. Thas

In 1997 J. A. Thas published the paper [6] in which a connection is made between good eggs in \( \text{PG}(4n-1, q), q \) odd, and Veronese Varieties in \( \text{PG}(5, q^n) \). Here we state the updated version of the main theorem as in [7], but leaving out the connection with translation generalized quadrangles.

**Theorem 1** (From Thas [7, Theorem 9.1])

*If the egg \( E \) of \( \text{PG}(4n-1, q), q \) odd, is good at an element \( E \), then we have one of the following.*

(a) There exists a \( \text{PG}(3, q^n) \) in the extension \( \text{PG}(4n-1, q^n) \) of \( \text{PG}(4n-1, q) \) which has exactly one point in common with each of the extensions of the egg elements. The set of these \( q^{2n} + 1 \) points is an elliptic quadric of \( \text{PG}(3, q^n) \) and \( E \) is elementary.

(b) We are not in case (a) and there exists a \( \text{PG}(4, q^n) \) in \( \text{PG}(4n-1, q^n) \) which intersects the extension of \( E \) in a line \( M \) and which has exactly one point \( r_i \) in common with the extension of the other egg elements. Let \( W \) be the set of these intersection points \( r_i \), \( i = 1, \ldots, q^{2n} \), and let \( M \) be the set of all common points of \( M \) and the conics which contain exactly \( q^n \) points of \( W \). Then the set \( W \cup M \) is the projection of a quadric Veronesean \( V_2^4 \) from a point \( P \) in a conic plane of \( V_2^4 \) onto \( \text{PG}(4, q^n) \); the point \( P \) is an exterior point of the conic of \( V_2^4 \) in the conic plane. In this case the egg \( E \) is isomorphic to the egg of Kantor-type.

(c) We are in case neither (a) nor (b) and there exists a \( \text{PG}(5, q^n) \) in \( \text{PG}(4n-1, q^n) \) which intersects the extension of \( E \) in a plane \( \pi \), and which has exactly one point \( r_i \) in common with the extension of the other egg elements. Let \( W \) be the set of these intersection points \( r_i \), \( i = 1, \ldots, q^{2n} \), and let \( P \) be the set of all common points of \( \pi \) and the conics which contain exactly \( q^n \) points of \( W \). Then the set \( W \cup P \) is a quadric Veronesean in \( \text{PG}(5, q^n) \).
Denote the egg elements by \( \{ E, E_1, \ldots, E_{q^{2n}} \} \) and let \( E \) be the good element. If we project the egg elements from one of its elements onto a \((3n - 1)\)-space \( \text{PG}(3n - 1, q) \) skew to that element then we obtain a partial \((n - 1)\)-spread of size \( q^{2n} \). If we project from \( E \) then we can extend this partial \((n - 1)\)-spread to a Desarguesian \((n - 1)\)-spread. It was proved by Segre [4] (see also [3]) that this implies that there exists an imaginary plane \( \pi \) in \( \text{PG}(4n - 1, q^n) \)

Let \( X \) be an element \( X \) of \( \text{PG}(3n - 1, q) \), where \( \sigma \) is the non-identity collineation of \( \text{PG}(4n - 1, q^n) \) fixing \( \text{PG}(4n - 1, q) \) pointwise. Let \( \rho \) be the \((n + 2)\)-space spanned by the good element and \( \pi \), and let \( P_i \) be the intersection of the extension of the egg element \( E_i \) with \( \rho \) (note that this intersection is indeed a point). Let \( \mathcal{W} = \{ P_i | i = 1, \ldots, q^{2n} \} \). Then we will show that one of the following cases occurs.

(a) \( \mathcal{W} \) generates a 3-space and then the egg is elementary.

(b) \( \mathcal{W} \) generates a 4-space and the egg is of Kantor-type (and \( \mathcal{W} \) is the affine part of a projection of a Veronesean of \( \text{PG}(5, q^n) \))

(c) \( \mathcal{W} \) generates a 5-space and \( \mathcal{W} \) is the affine part of a Veronesean in \( \text{PG}(5, q^n) \).

3 A new proof

In this section we give a proof of Theorem 1 using the model for good eggs given in the introduction. Let \( V(n, q^n) \) denote an \( n \)-dimensional vectorspace over \( \text{GF}(q^n) \). Let \( V(n, q) \) be the vectorspace consisting of vectors of \( V(n, q^n) \) with coordinates in \( \text{GF}(q) \) with respect to a fixed basis. The egg elements are represented as subsets of \( \text{PG}(3, q^n) \). In order to write down the extension of the egg elements to subspaces of \( \text{PG}(4n - 1, q^n) \), we construct a suitable embedding of \( \text{GF}(q^n) \) in \( V(n, q^n) \) in the following way. Let \( X \) be an element of \( \text{GL}(n, q) \) of order \( q^n - 1 \), and let \( \mathbf{v} \) be an eigenvector of \( X \) with eigenvalue \( \lambda \). Then \( \lambda \) is a primitive element of \( \text{GF}(q^n) \), and the eigenvalues of \( X \) are \( \lambda, \lambda^2, \ldots, \lambda^{q^n-1} \) with corresponding eigenvectors \( \mathbf{v}, \mathbf{v}\sigma, \ldots, \mathbf{v}\sigma^{n-1} \), where \( \sigma : (x_1, \ldots, x_n) \mapsto (x_1^2, \ldots, x_n^q) \). And so with every \( \alpha \in \text{GF}(q^n)^* \) there corresponds a certain power of \( X \), which we denote by \( Z(\alpha) \), such that \( \alpha \mathbf{v} = \mathbf{v} Z(\alpha) \). Define \( \mathbf{e}_{k+1} = \lambda^k \mathbf{v} + \lambda^{kq} \mathbf{v}\sigma + \cdots + \lambda^{kq^{n-1}} \mathbf{v}\sigma^{n-1} \), for \( k = 0, \ldots, n - 1 \). Then \( \{ \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \} \) is a basis of \( V(n, q^n) \), consisting of vectors of \( V(n, q) \), since \( \{ \mathbf{v}, \mathbf{v}\sigma, \ldots, \mathbf{v}\sigma^{n-1} \} \) is a basis for \( V(n, q^n) \). We define the bijection

\[
\alpha = a_1 + a_2 \lambda + \cdots + a_n \lambda^{n-1} \mapsto \tilde{\alpha} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + \cdots + a_n \mathbf{e}_n
\]
between $\text{GF}(q^n)$ and $V(n, q)$. Since
\[
\mathbf{e}_1 Z(a_1 + a_2 \lambda + \cdots + a_n \lambda^{n-1}) = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + \cdots + a_n \mathbf{e}_n,
\]
we have $\overline{\alpha \beta} = \mathbf{e}_1 Z(\alpha \beta) = \mathbf{e}_1 Z(\beta)Z(\alpha) = \overline{\beta} Z(\alpha)$. This implies that $Z(\alpha)$ is the matrix of the linear transformation in $V(n, q)$ corresponding to multiplying by $\alpha$ in $\text{GF}(q^n)$. The automorphism $\alpha \mapsto \alpha^q$ of $\text{GF}(q^n)$ defines the $\text{GF}(q)$—semilinear map $A$ from $V(n, q)$ in itself, such that $Z(\alpha^q) = A^{-1} Z(\alpha) A$. This implies that $\mathbf{v} A^{-1} Z(\alpha) A = \alpha^q \mathbf{v}$, i.e., $\mathbf{v} A^{-1}$ is an eigenvector of $Z(\alpha)$ with eigenvalue $\alpha^q$, and it follows that $\mathbf{v} A^{-i}$ is an eigenvector of $Z(\alpha)$ with eigenvalue $\alpha^q$. We identify the $\text{GF}(q)$—linear map $t \mapsto g_t(a, b) = \sum_{i=0}^{n-1} (\gamma_i t)^{1/q^i}$ (with $\gamma_i = a_i a^2 + b_i a b + c_i b^2$) in $\text{GF}(q^n)$ with the $\text{GF}(q)$—linear map $L_{a,b}$ in $V(n, q^n)$ defined by: $L_{a,b}(\alpha) = \beta$ if and only if $g_{a_b}(a, b) = \beta$, for all $\alpha, \beta \in \text{GF}(q^n)$. Hence $L_{a,b} : a \mapsto \sum_{i=0}^{n-1} \alpha A^{-i} Z(\gamma_i^{1/q^i})$, and $L_{a,b}(\mathbf{v}) = \sum_{i=0}^{n-1} \gamma_i \mathbf{v} A^{-i}$. Now we can write down the extension of the egg elements as
\[
\overline{E}(a, b) = \langle -L_{a,b}(\mathbf{w}), \mathbf{w}, \mathbf{w} Z(-a), \mathbf{w} Z(-b) \rangle \| \mathbf{w} \in V(n, q^n) \rangle,
\]
for all $a, b \in \text{GF}(q^n)$. By projecting the egg elements from the good element onto $\text{PG}(3n - 1, q) = \{ (0, r, s, t) \| (r, s, t) \in (\text{GF}(q^n))^3 \}$, the Desarguesian spread obtained this way, corresponds to the imaginary plane $\pi$ generated by $(0, \mathbf{v}, 0, 0), (0, 0, \mathbf{v}, 0), (0, 0, 0, \mathbf{v})$. Eventually we find that the point $P_1$ (with $E_i = E(a, b)$) has coordinates $(-L_{a,b}(\mathbf{v}), \mathbf{v}, -a \mathbf{v}, -b \mathbf{v})$, and hence
\[
\mathcal{W} = \{ (-\sum_{i=0}^{n-1} (a_i a^2 + b_i a b + c_i b^2) \mathbf{v} A^{-i}, \mathbf{v}, -a \mathbf{v}, -b \mathbf{v}) \| a, b \in \text{GF}(q^n) \}.
\]
If $(a_0, a_1, \ldots, a_{n-1})$ or $(c_0, c_1, \ldots, c_{n-1})$ is $0$ then one easily sees that the tangent space at $E(\infty)$ intersects one of the other egg elements, contradicting the definition of an egg. If $(b_0, b_1, \ldots, b_{n-1}) \neq 0$, then $\mathcal{W}$ is contained in the subspace spanned by $Q_1 := (-\sum_{i=0}^{n-1} a_i \mathbf{v} A^{-i}, 0, 0, 0), Q_2 := (-\sum_{i=0}^{n-1} b_i \mathbf{v} A^{-i}, 0, 0, 0), Q_3 := (-\sum_{i=0}^{n-1} c_i \mathbf{v} A^{-i}, 0, 0, 0), Q_4 := (0, \mathbf{v}, 0, 0), Q_5 := (0, 0, -\mathbf{v}, 0)$, and $Q_6 := (0, 0, 0, -\mathbf{v})$. If the dimension of $U := \langle Q_1, Q_2, Q_3, Q_4, Q_5, Q_6 \rangle$ is 5 then by taking $\{ Q_1, Q_2, Q_3, Q_4, Q_5, Q_6 \}$ as a basis for $U$ the points of $\mathcal{W}$ have coordinates $\langle a^2, ab, b^2, 1, a, b \rangle$ $\langle a_0, a_1, \ldots, a_{n-1} \rangle \in \text{GF}(q^n)$. This is the affine part of the Veronesean $\mathcal{V}_2$ of degree 4, and this proves part (c) of the theorem. If $U$ has dimension 4 then, since we may assume $(a_0, a_1, \ldots, a_{n-1}) = (1, 0, \ldots, 0)$ (see e.g. Remark 1.3 in [1]), there exists a $\gamma \in \text{GF}(q^n)^*$ such that $(b_1, b_2, \ldots, b_{n-1}) = \gamma(c_1, c_2, \ldots, c_{n-1})$. But then it is a straightforward calculation to see that the tangent space at $E(0, 1)$ contains an element of the Desarguesian spread induced by the $q^n + 1$ egg elements contained in $\langle E(\infty), E(0, 0), E(1, 0) \rangle$ (see e.g. the proof of Theorem 4.2 in [1]). By [1, Theorem 4.1] we may conclude that the egg is of Kantor type. Choosing $\{ Q_1, Q_2, Q_4, Q_5, Q_6 \}$ as a basis for $U$, we see that $\mathcal{W}$ is a projection of the Veronesean from the point $(0, -\gamma, 1, 0, 0, 0)$ onto the the hyperplane with equation $X_2 = 0$. This proves part (b). If $U$ is 3-dimensional then again with $(a_0, a_1, \ldots, a_{n-1}) = (1, 0, \ldots, 0), b_i = c_i = 0$ for
It is easy to see that then the egg is elementary and the set $W$ is the set of affine points on an elliptic quadric. If $(b_0, b_1, \ldots, b_{n-1}) = \mathbf{0}$, then by using completely the same arguments as above one easily sees that the egg is either of Kantor type or elementary.

References

[1] Michel Lavrauw; Characterisations and properties of good eggs in $\text{PG}(4n - 1, q)$, $q$ odd, to appear in *Discrete Math*.


