The two sets of three semifields associated with a semifield flock

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Abstract

In 1965 Knuth [4] showed that from a given finite semifield one can construct further semifields manipulating the corresponding cubical array, and obtain in total six semifields from the given one. In the case of a rank two commutative semifield (the semifields corresponding to a semifield flock) these semifields have been investigated in [1], providing a geometric connection between these six semifields and it was shown that they give at most three non-isotopic semifields. However, there is another set of three semifields arising in a different way from a semifield flock, hence in total six semifields arise from a rank two commutative semifield (see [1]). In this article we give a geometrical link between these two sets of three semifields.

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1 Introduction and motivation

Throughout the article we will use the terminology and the notation from [1]. A semifield coordinatises a semifield plane, which corresponds to a semifield spread via the Andre-Bruck-Bose construction, see [3, Section 3.1]. A flock of a quadratic cone gives rise to a line spread of three-dimensional projective space (and hence to a translation plane) via the Thas-Walker construction, see [1], [6]. In case the flock is a semifield flock, the resulting translation plane is a semifield plane.

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Remark 1.1. Two semifield planes are isomorphic if and only if the corresponding semifields are isotopic. Usually we are only interested in the number of non-isomorphic planes corresponding to a semifield plane and hence in the number of isotopy-classes arising from a semifield. Out of convenience we will often talk about the number of semifields, (for instance in the title) instead of the number of isotopy-classes of semifields.

Starting with a semifield flock, one can also construct a rank two commutative semifield, by coordinatising the projective space of the flock, in order to obtain a so-called Cohen-Ganley pair of functions \((f, g)\). Following [1], let \(S\) denote the semifield obtained from a semifield flock using the Thas-Walker construction. As shown in [1] the six semifields associated to \(S\) under the Knuth-operations give three isotopy classes of semifields, which can geometrically be generated dualising the semifield plane \(S \mapsto S^*\) and dualising the semifield spread \(S \mapsto S^\dagger\). The three isotopy classes can be represented by \(S \cong S^!, S^*, S^\dagger \cong S^*\dagger\). The rank two commutative semifield is \(\hat{S}^*\dagger\), where \(\hat{S}\) is the semifield corresponding to the symplectic spread, arising from the translation ovoid of \(Q(4, q)\) associated to the semifield flock. As shown in [1] the six semifields associated to \(\hat{S}\) under the Knuth-operations give three isotopy classes of semifields, which can be represented by \(\hat{S} \cong \hat{S}^!, \hat{S}^*, \hat{S}^\dagger \cong \hat{S}^*\dagger\). Here we provide a geometric link for the operation \(S \mapsto \hat{S}\).

2 Dualising the ovoid of the Klein quadric

The key idea is to use a particular representation of the Klein quadric due to Lunardon [5], denoted by \(T_4(Q^+(3, q^n))\) and construct the translation dual (see [5]) of the translation ovoid. First we introduce some notation. If \(r(x)\) is a linearized \(q\)-polynomial over \(\text{GF}(q^n)\), i.e.,
\[
r(x) = \sum_{i=0}^{n-1} r_i x^{q^i},
\]
for some \(r_i \in \text{GF}(q^n)\), then we define \(\hat{r}(x)\) by
\[
\hat{r}(x) = \sum_{i=0}^{n-1} (r_i x)^{1/q^i}.
\]
Consider the pre-semifield of rank two over its left nucleus \(\text{GF}(q^n)\) with multiplication
\[
(x, y) \circ (u, v) = (xf(v) + yu + yg(v), xu + yv),
\]
where \( f \) and \( g \) are linearized \( q \)-polynomials in \( \text{GF}(q^n)[X] \) as in \([1]\), satisfying the conditions for a so called Cohen-Ganley pair that \( g(x)^2 + 4xf(x) \) is a non-square for all \( x \in \text{GF}(q^n)^* \). The corresponding spread set is

\[
\left\{ \begin{pmatrix} f(v) & u \\ u + g(v) & v \end{pmatrix} \mid u, v \in \text{GF}(q^n) \right\}.
\]

**Remark 2.1.** As mentioned before, we are only interested in the number of isotopy classes of semifields, and hence we need to choose a representative of each class. (Ideally we would like to have a canonical form for each isotopy class.) The multiplications listed in \([1, \text{Table 1}]\) are often corresponding to a pre-semifield instead of a semifield. In Section 3 we provide a table representing the six isotopy classes, such that each multiplication corresponds to a semifield with \((1, 0)\) as unit element.

Following the above remark, we will continue with the spread set

\[
\left\{ \begin{pmatrix} u & v \\ f(v) & u + g(v) \end{pmatrix} \mid u, v \in \text{GF}(q^n) \right\}.
\]

Note that the condition for this to be a spread set is the same as before. The corresponding multiplication in the semifield is

\[
(x, y) \circ (u, v) = (ux + yf(v), xv + yu + yg(v)).
\]

Since \( f(0) = g(0) = 0 \), it immediately follows that \((1, 0)\) is the unit element in this semifield. The corresponding ovoid \( \mathcal{O} \) of \( Q^+(5, q^n) : X_0X_5 + X_1X_4 + X_2X_3 = 0 \) is the set of points

\[
\langle 1, u, v, -f(v), u + g(v), vf(v) - u^2 - ug(v) \rangle, \ u, v \in \text{GF}(q^n),
\]

and the point \( \langle 0, 0, 0, 1 \rangle \). By looking at \( Q^+(5, q^n) \) as \( T_4(Q^+(3, q^n)) \) (see \([5]\)) we obtain the \((2n - 1)\)-space

\[
U = \{ (0, u, v, -f(v), u + g(v), 0) \mid (u, v) \in (\text{GF}(q^n)^2)^* \}
\]

over \( \text{GF}(q) \) skew from \( Q^+(3, q^n) \) with equation \( X_1X_4 + X_2X_3 = 0 \) in the three-dimensional space with equation \( X_0 = X_5 = 0 \). Note that the condition for \( U \) to be skew from \( Q^+(3, q^n) \) is exactly the condition for the set of matrices to be a spread set. Dualising with respect to the duality defined by the bilinear form over \( \text{GF}(q) \)

\[
(a, b) = \text{tr}(a_1b_1 + a_4b_1 + a_2b_3 + a_3b_2),
\]

where \( \text{tr}(x) = \sum_{i=0}^{n-1} x^{q^i} \) we obtain the \((2n - 1)\)-space skew from \( Q^+(3, q^n) \) inducing again a translation ovoid of \( Q^+(5, q^n) \). When calculating the dual space of \( U \) one sees that \( U^D \) consists of points \( \langle 0, x, y, z, w, 0 \rangle \) for which

\[
\text{tr}(x(u + g(v)) - yf(v) + zv + wu) = 0, \ \forall u, v \in \text{GF}(q^n).
\]
Putting \( v = 0 \) gives \( w = -x \), and putting \( u = 0 \) gives \( \text{tr}(xg(v) - yf(v) + zv) = 0 \), \( \forall v \in GF(q^n) \). This implies that \( z = -\hat{g}(x) + \hat{f}(y) \) (since \( \text{tr}(yr(x)) = \text{tr}(x\hat{r}(y)) \) for any \( q \)-linearized polynomial \( r \)) and we may conclude that

\[
U^D = \{ (0, x, y, -\hat{g}(x) + \hat{f}(y), -x, 0) \parallel (x, y) \in (GF(q^n)^2)^* \}.
\]

By construction \( U^D \) is skew from the quadric \( Q^+(3, q^n) \), and this is the exact same condition that \( -x^2 - y\hat{g}(x) + y\hat{f}(y) = 0 \) implies \( (x, y) = 0 \), as for the set of matrices

\[
\left\{ \begin{pmatrix} u & v \\ v & \hat{f}(u) - \hat{g}(v) \end{pmatrix} \parallel u, v \in GF(q^n) \right\},
\]

to be a spread set. The multiplication in the corresponding pre-semifield is

\[
(x, y)\hat{\circ}(u, v) = (xu + yv, xv + y\hat{f}(u) - y\hat{g}(v)).
\]

Let \( \hat{\pi} \) denote the semifield plane corresponding to the pre-semifield \( \hat{S} \) as in [1, Table 1].

**Theorem 2.2.** The semifield plane corresponding to the pre-semifield \( (GF(q^n)^2, +, \hat{\circ}) \) is isomorphic to the semifield plane \( \hat{\pi} \).

**Proof.** Let \( F(x, y) = (y, -x) \) and \( G(u, v) = (-v, u) \). Then

\[
F((x, y)\hat{\circ}(u, v)) = (xv + y\hat{f}(u) - y\hat{g}(v), -xu - yv)
\]

\[
= (y, -x) \cdot (-v, u) = F(x, y) \cdot G(u, v),
\]

where \( \cdot \) is the multiplication

\[
(x, y) \cdot (u, v) = (yu + x\hat{f}(v) + x\hat{g}(u), xu + yv)
\]

of the pre-semifield \( \hat{S} \) as in [1, Table 1]. This implies that the two pre-semifields are isotopic and hence that the two semifield planes are isomorphic. \( \square \)

We may conclude that apart from operation \( * \) (dualising the plane), operation \( \dag \) (dualising the spread), also the operation \( S \mapsto \hat{S} \) has a geometric interpretation (dualising the ovoid).

### 3 The six semifields associated to a semifield flock

In this section we provide a table with the semifield multiplication (instead of pre-semifield multiplication), with unit element \((1, 0)\), for each of the six isotopy classes of semifields corresponding to a semifield flock.
As in Section 2 let $S$ denote the semifield with multiplication

$$(x, y) \circ (u, v) = (ux + yf(v), xv + yu + yg(v)).$$

Dualising the plane we get the semifield $S^*$ by reversing the multiplication, i.e.,

$$(x, y) \circ^* (u, v) = (xu + vf(y), uy + xv + vg(y)).$$

Both multiplications have $(1, 0)$ as identity element. In order to obtain the multiplication for $S^{**}$ we have to dualise the semifield spread obtained from $S^*$ (see [1]). We have to find all $a, b, c, d \in GF(q^n)$ for which

$$\text{tr}(xa + yb + (xu + f(y)v)c + (yu + xv + g(y)v)d) = 0, \forall x, y \in GF(q^n).$$

Putting $x = 0$ we get the condition

$$\text{tr}(yb + f(y)vc + yu + g(y)vd) = 0, \forall y \in GF(q^n).$$

This implies $b = -(\hat{f}(vc) + ud + \hat{g}(vd))$. Similarly, after putting $y = 0$ we get $a = -uc - vd$. Hence after some coordinate transformations, we get the multiplication

$$(x, y) \cdot (u, v) = (xu + yf(v), xv + yf(v) + \hat{g}(uv))$$

In order for $(1, 0) = (1, 0) \cdot (1, 0)$ to be the identity we have to define a new multiplication by $((x, y) \cdot (1, 0)) \circ^* ((1, 0) \cdot (u, v)) = (x, y) \cdot (u, v)$ (see [4]). We get

$$(x, y) \circ^* (u, v) = (xu + y\hat{f}^{-1}(v), uy + \hat{f}(x\hat{f}^{-1}(v)) + \hat{g}(y\hat{f}^{-1}(v))).$$

That $\hat{f}^{-1}$ is well defined follows from the fact that the multiplication $\circ$ from the previous section has no zero divisors. In the previous we had the following multiplication for $\hat{S}$:

$$(x, y)\circ(u, v) = (xu + yv, xv + y\hat{f}(u) - y\hat{g}(v)).$$

We see that $(1, 0)\circ(u, v) = (u, v)$ and, $(x, y)\circ(1, 0) = (x, y\hat{f}(1))$, and in order for $(1, 0)$ to be the identity, we can apply one of the methods to get a semifield from a pre-semifield (see [4]) and define a new multiplication. We use the same notation $\hat{S}$ for the semifield with identity $(1, 0)$ and multiplication

$$(x, y)\circ(u, v) = (xu + y\hat{f}^{-1}(1)v, xv + y\hat{f}^{-1}(1)\hat{f}(u) - y\hat{f}^{-1}(1)\hat{g}(v)).$$

Reversing this multiplication we get the semifield $\hat{S}^*$, i.e.,

$$(x, y)\circ^*(u, v) = (xu + y\hat{f}^{-1}(1)v, yu + v\hat{f}^{-1}(1)\hat{f}(x) - v\hat{f}^{-1}(1)\hat{g}(y)).$$
Table 1: The six semifield multiplications with identity \((1,0)\), defined on the set of elements of \(\text{GF}(q^{n})^{2}\) (addition as in \(\text{GF}(q^{n})^{2}\)) associated with a semifield flock. The nuclei are as in [1] with \(q\) replaced by \(q^{n}\) and \(q_{0}\) replaced by \(q^{n}\).

\[
\begin{align*}
S & (x, y) \circ (u, v) = (ux + yf(v), xv + yu + yg(v)) \\
S^{*} & (x, y) \circ^{*} (u, v) = (xu + ef(y), uy + xv + vq(y)) \\
S^{*\dagger} & (x, y) \circ^{*\dagger} (u, v) = (xu + yf^{-1}(v), uy + xv + gf(v)) \\
\check{S} & (x, y) \check{\circ} (u, v) = (xu + yf^{-1}(v), xv + yf^{-1}(1)f(u) - yf^{-1}(1)g(v)) \\
\check{S}^{*} & (x, y) \check{\circ}^{*} (u, v) = (xu + yf^{-1}(1)v, xv + yf^{-1}(1)f(x) - vf^{-1}(1)g(y)) \\
\check{S}^{*\dagger} & (x, y) \check{\circ}^{*\dagger} (u, v) = (xu + f(yv), xv + yu - g(yv))
\end{align*}
\]

Finally we get the semifield \(\check{S}^{*\dagger}\) by dualising the semifield spread corresponding to \(\check{S}^{*}\). As before, after applying the same methods in order to obtain a multiplication with identity \((1,0)\), we get

\[
(x, y) \check{\circ}^{*\dagger} (u, v) = (xu + f(yv), xv + yu - g(yv)).
\]

The following table summarizes these results.

**Remark 3.1.** Note that this operation (dualising the ovoid) can be extended to all finite semifields which are of rank two over their left nucleus (and so corresponding to spreads of \(\text{PG}(3, q^{n})\) and hence ovoids of \(Q(5, q^{n})\)). In fact this turns out to be a special case of one of the semifield operations from [2].

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**References**


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