Asymptotic Geometry and Topology of Random Real Algebraic Hypersurfaces

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December 9, 2020
We let $\mathbb{RP}^m$ denote the real projective space that is obtained by identifying the points $X, Y \in \mathbb{R}^{m+1} \setminus \{0\}$ iff $X = \lambda Y$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. Then $\mathbb{RP}^m$ is defined to be the set of all equivalence classes. We also let $[x_0, x_1, \ldots, x_m]$ be the homogenous coordinates on $\mathbb{RP}^m$. There is a natural projection

$$\pi : \mathbb{R}^{m+1} \setminus \{0\} \rightarrow \mathbb{RP}^m$$

$$\pi(x_0, x_1, \ldots, x_m) = [x_0, x_1, \ldots, x_m].$$

One can similarly define the complex version $\mathbb{CP}^m$ complex projective space by identifying the points of a complex line in $\mathbb{C}^{m+1}$ passing through the origin.
Motivation:

Let $W(m, n)$ denote the real vector space of homogenous polynomials in $m + 1$ variables of degree $n$.

**Example**

Let $P(x, y, z) = 2x^n + x^2y^{n-2} + 3yz^{n-1} \in W(2, n)$ for an integer $n \geq 2$. 
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For $f \in W(m, n)$ we denote the corresponding real and complex algebraic hypersurfaces by

$$V_f := \{x \in \mathbb{RP}^m | f(x) = 0\}$$

$$Z_f := \{z \in \mathbb{CP}^m | f(z) = 0\}.$$

**Objective:** We are interested in geometric (e.g. projective volume $Vol(V_f)$) and topological features (e.g. # of connected components $N(V_f)$ and their distribution) of “generic” polynomial as the degree $n \to \infty$. 
Motivation: Dimension one

In dimension one (i.e. \( m = 1 \)), for \( f(x) \in \mathbb{R}[x] \) the first problem corresponds to counting real zeros of polynomials and the second one is studying asymptotic distribution of the real zeros.

Everyone knows the following

\[
0 \leq \#V_f \leq \#Z_f = \text{deg}(f) = n \quad \text{and} \quad \#V_f \equiv n \pmod{2}.
\]

Eg. In general anything is possible: \( P(x) = x^2 + 1 \) has no real roots whereas \( Q(x) = x^2 - 1 \) has two real roots in \( \mathbb{R} \).

**Question**

How many real roots can a “generic” polynomial have?
Idea: We endow $W(m, n)$ with a probability measure (to be defined). This allows us to regard

$$Vol(V_f), N_n(V_f) : (W(m, n), Prob_n) \rightarrow \mathbb{N}$$

as random variables and we wish to study their statistics (their mean and variance etc.)

1. Kac-Rice Formula for the \# of real zeros and applications
2. Higher Dimensions: Asymptotic growth of
   - \# of connected components and their distribution
   - projective volume
3. Brief discussion of proofs
4. Further directions and open problems
Kac-Rice Formula

Let $f_0(t) \equiv 1$, $f_1(t)$, $f_2(t)$, ..., $f_n(t) : \mathbb{R} \rightarrow \mathbb{R}$ be a sequence of $C^1$ functions. We consider the Gaussian process

$$F(t) := \sum_{j=1}^{n} a_j f_j(t)$$

where $a_j$ are independent copies of standard real Gaussian $N(0, 1)$. For any interval $I \subset \mathbb{R}$ we define the random variable

$$N(F, I) := \# \{ t \in I : F(t) = 0 \}.$$ 

We also denote the **covariance matrix** of $F(t)$ and $F'(t)$ is defined by

$$\Sigma_t := \begin{bmatrix} \alpha_t & \beta_t \\ \beta_t & \gamma_t \end{bmatrix} = \begin{bmatrix} \mathbb{E}[(F(t))^2] & \mathbb{E}[F(t)F'(t)] \\ \mathbb{E}[F(t)F'(t)] & \mathbb{E}[(F'(t))^2] \end{bmatrix}.$$
Theorem (M. Kac '43)

Assume that

- For a.e. \((a_0, \ldots, a_n)\) the function \(F(t)\) does not have a double root
- \# of zeros + \# of critical pts \(\leq C\) for some uniform constant \(C > 0\)
- Then covariance matrix \(\Sigma_t\) is non-singular for each \(t \in I\)
- \(\sqrt{\det \Sigma_t} + \beta_t\) is integrable over \(I\).

Then

\[
\mathbb{E}[N(F, I)] = \int_I \mathbb{E}[F'(t) | F(t) = 0] \rho_F(0) dt
\]

where \(\rho_F(0) = \frac{1}{\sqrt{2\pi} \sqrt{\langle \nu(t), \nu(t) \rangle}}\) and \(\nu(t) := (1, f_1(t), \ldots, f_n(t))\).
Kac-Rice Formula

A standard calculation shows that

$$\mathbb{E}[N(F, I)] = \frac{1}{\pi} \int_I \frac{\sqrt{\det \Sigma_t}}{\alpha_t} dt$$

$$= \frac{1}{\pi} \int_I \sqrt{\frac{\partial^2}{\partial x \partial y} \log \langle v(x), v(y) \rangle \bigg|_{x=y=t}} dt$$

**Corollary (Kac)**

Let $a_j$ are independent copies of standard real Gaussian $N(0,1)$ and $F(t) = \sum_{j=0}^n a_j t^j$ be a Kac polynomial. Then

$$\mathbb{E}[N(F, \mathbb{R})] = \frac{2}{\pi} \log n + o(1)$$

as $n \to \infty$. 
Proof of Kac’s Formula (Edelman-Kostlan ('95) Approach)

Let us denote by \( \vec{a} = (a_0, \ldots, a_n) \), \( \vec{u} = \frac{\vec{a}}{\|\vec{a}\|} \) and \( \gamma(t) = \frac{v(t)}{\|v(t)\|} \) where \( v(t) = (1, t, t^2, \ldots, t^n) \). Note that \( F(t) = 0 \iff \vec{u} \perp \gamma(t) \). Since \( \vec{u} \) is uniformly distributed on the sphere \( \mathbb{S}^n \) with respect to surface area measure. Hence, we obtain

\[
\mathbb{E}[N(F, I)] = \mathbb{E}_u[\#(u^\perp \cap \gamma)] = \frac{1}{\pi} |\gamma|
\]

where \( |\gamma| \) denotes the length of the curve \( \gamma = \{\gamma(t) | t \in I\} \subset \mathbb{S}^n \). The last equality is known as Crofton’s formula in Integral Geometry. Now, by calculus

\[
|\gamma| = \int_I \|\gamma'(t)\| dt
\]

and using \( \|\gamma'(t)\|^2 = \frac{\partial^2}{\partial x \partial y} \log \langle v(x), v(y) \rangle |_{x=y=t} \) we obtain Kac’s formula.
Example (E-K '95, SU(2) Polynomials)

Consider random polynomials

\[ F(t) = \sum_{j=0}^{n} a_j \sqrt{\binom{n}{j}} t^j \]

where \( a_j \) are independent copies of \( N(0, 1) \). Then

\[ \mathbb{E}[N(F, \mathbb{R})] = \sqrt{n}. \]

Proof.

Indeed, by Binomial formula \( \langle v(x), v(y) \rangle = (1 + xy)^n \) and

\[ \mathbb{E}[N(F, \mathbb{R})] = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sqrt{n}}{1 + t^2} dt = \sqrt{n} \]
We also remark that \( f_j(z) = \sqrt{\binom{n}{j}} z^j \) form an ONB with respect to the inner product

\[
\langle f, g \rangle_n := \int_{\mathbb{C}} f(z) \overline{g(z)} \frac{dz}{(1 + |z|^2)^{n+2}}
\]

This implies that the corresponding Bergman kernel (i.e. the integral kernel of the orthogonal projection)

\[
L^2(\mathbb{C}, \frac{dz}{(1 + |z|^2)^2}) \rightarrow \text{Poly}_n
\]

is given by

\[
K_n(z, w) = \langle \nu(z), \nu(w) \rangle.
\]

In particular, computing the expected number of real roots boils down to near and off diagonal asymptotics of the Bergman kernel and its partial derivatives.
Let $Q : \mathbb{C} \to \mathbb{R}$ be a $C^2$ weight function such that $Q(z) = Q(|z|)$ for $z \in \mathbb{C}$ and $Q(z) \geq (1 + \epsilon) \log |z|$ for $|z| \gg 1$. Then we define

$$\langle f, g \rangle := \int_{\mathbb{C}} f(z)\overline{g(z)} e^{-2nQ(z)} dz$$

For a fixed ONB $\{f^n_j\}_j$ as before we define a random polynomial by

$$F(z) := \sum_j a^n_j f^n_j(z) \text{ where } a^n_j \text{ are i.i.d. r.v.}$$

**Theorem (B. ’18, Potential Anal.)**

Let $a^n_j$ be i.i.d. r.v. of mean zero and variance one such that $\mathbb{E}[|a^n_j|^{2+\epsilon}] < \infty$. Then

$$\mathbb{E}[N(F, R)] = \frac{\sqrt{n}}{\pi} \left( \int_{B_Q \cap \mathbb{R}} \sqrt{\frac{1}{2}} \Delta Q(x) dx + o(1) \right)$$

as $n \to \infty$ where $B_Q := \{z \in S_Q | \Delta Q(z) > 0\}$. 
We wish to generalize the aforementioned results to higher dimensions. There are at least two natural ways to do this: we may compute the asymptotic growth of the projective volume $Vol(V_f)$ or we may consider number of connected components $N(V_f)$.

First, we endow $W(m, n)$ with a centered Gaussian probability measure: let $\langle , \rangle$ be a real inner product on $W(m, n)$. Then for $A \subset W(m, n)$ define

$$Prob_n(A) := \frac{1}{c_{m,n}} \int_A e^{-\frac{\|F\|^2}{2}} dF$$

where $\|F\|^2 = \langle F, F \rangle$ and $c_{m,n}$ is a normalizing constant.

We remark that this definition is equivalent to the one appeared in previous slides.
Let \( f(x) = \sum_{|J|=n} c_J x^J \) and \( g(x) = \sum_{|J|=n} d_J x^J \) be two homogenous polynomial in \( m + 1 \) variables. We define

1. **(Kac Ensemble)** \( \langle f, g \rangle = \sum_{|J|=n} c_J d_J \)

2. **(Complex Fubini-Study Ensemble)** \( \langle f, g \rangle_C = \sum_{|J|=n} c_J d_J \binom{n}{|J|} \)
   where \( \binom{n}{|J|} = \frac{n!}{j_0! \cdots j_m!} \). We remark that up to a constant

   \[
   \langle f, g \rangle_C = \int_{\mathbb{C}^{m+1}} f(z) \overline{g(z)} e^{-\|z\|^2} \, dm(z) = \int_{\mathbb{CP}^m} f(\zeta) \overline{g(\zeta)} \, d\sigma_{FS}(\zeta).
   \]

3. **(Real Fubini-Study Ensemble)**

   \[
   \langle f, g \rangle_R = \int_{\mathbb{R}^{m+1}} f(x) g(x) e^{-\|x\|^2} \, dm(x) = \int_{\mathbb{RP}^m} f(x) g(x) \, d\sigma_{RFS}(x) \]

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   Random Real Algebraic Geometry
For $f \in W(2, n)$ we say that $V_f$ is non-singular if the gradient vector $\nabla(f)$ does not vanish at any point on $\text{Cone}(V_f) = \pi^{-1}(V_f)$, where $\pi : \mathbb{R}^3 - \{0\} \to \mathbb{RP}^2$.

It is a basic result that, when non-singular, $V_f$ is homeomorphic to a disjoint union of finitely many copies of $S^1$. Furthermore, if $n$ is odd, then precisely one of these components (a pseudoline) realizes a nontrivial class in $H_1(\mathbb{RP}^2) \cong \mathbb{Z}/2\mathbb{Z}$ and all of the other components are homologically trivial (ovals). If, on the other hand, $n$ is even then all components are ovals.

**Problem (Hilbert’s Sixteenth Problem)**

Broadly speaking, classify topology of nonsingular real algebraic curves: study of the maximal number and the possible arrangements of the components.
Expected Topology: Real Algebraic Plane Curves

Recall that if the curve has no singularities (real or complex) then $Z_f \subset \mathbb{CP}^2$ is a Riemann surface of genus

$$g = (n - 1)(n - 2)/2.$$ 

Then we have the following classical result

**Theorem (Harnack)**

The number of connected components of a non-singular real algebraic curve is bounded above by $g + 1$.

The upper bound can be realized for each value of $n$.

**Definition**

Non-singular algebraic curves realizing this upper bound are called **M-curves**.
For $n \in \mathbb{N}^*$ and $q \in \mathbb{Q}_+^*$ let

$$M_{n,q} := \{ f \in W(2,n) : N(V_f) \geq g + 1 - nq \}.$$

The set $M_{n,q}$ is always non-empty for sufficiently large $n$. Moreover, Gayet and Welschinger showed that the probability of it decreases exponentially fast:

**Theorem (Gayet-Welschinger, Publ. IHES (2011))**

We consider $W(2,n)$ endowed with the complex Fubini-Study probability measure. Then for every sequence $q_n \geq 1$ of rationals there exists constants $C, D > 0$ such that

$$\text{Prob}(M_{n,q_n}) \leq Cq_n^4 \exp\left(-D \frac{n}{q_n}\right).$$
Sketch of the proof of GW Theorem

The upper bound on the probability $\text{Prob}(M_{n,q_n})$ is a nice application of the theory of laminar positive closed currents, a widely used concept in complex dynamics. More precisely, GW observes that if the real locus $V_f$ has many components, then for any complex closed ball $B \subset \mathbb{CP}^2 \setminus \mathbb{RP}^2$ the genus $g(Z_{f_n} \cap B)$ can be bound by its area.

Let $f_n \in W(2, n)$ be a sequence of such “exceptional” polynomials. Then by a Theorem of H. de Thelin [Ann. Sci. Ecole Norm. Sup. (4) 37 (2004)] any cluster current of the sequence $\frac{1}{n}[Z_{f_n}]$ of normalized integration currents on $Z_{f_n}$ admits a laminar structure outside $\mathbb{RP}^2$. However, it is known that [Shiffman-Zelditch, Comm. Math. Phys. (1999)] normalized integration currents $\frac{1}{n}[Z_{f_n}]$ uniformly distributed with respect to the Fubini-Study form $\omega_{FS}$ on $\mathbb{CP}^2$. This implies the result.
In a recent joint work with Ö. Kişisel (METU), we addressed a slightly different question:

**Question**

*Given a point $p \in \mathbb{RP}^2$ what is the expected number of components (of a random real algebraic curve) that are winding around $p$?*

More precisely, every oval subdivides $\mathbb{RP}^2$ into two connected components, one homeomorphic to a disk, the other homeomorphic to a Möbius band. Let the disk component be called the *interior* of that oval and be denoted by $\text{int}(K)$. For a point $p$ in $\mathbb{RP}^2$ we wish to give a formula for the expected number of ovals that contains $p$ in its interior.
Here is the hypothetical picture of a random real algebraic curve in the plane:

Since any number of ovals that contain the same point $p$ in their interior are necessarily nested. We define the random variable

$$\ell_p : (W_{2,n}, Prob_n) \rightarrow \mathbb{N}$$

and we wish to compute its expected value $\mathbb{E}[\ell_p]$. 
We identify the point $p$ with the origin $(0, 0)$ in the affine chart 
$\{[X : Y : Z] \in \mathbb{RP}^2 : Z \neq 0\}$ and assume that the equation of $V_f$
in the affine chart is given by $f(x, y) = 0$ where 

$$f(x, y) = \sum_{|J| \leq n} a_J c_J x^{j_1} y^{j_2}$$

$a_J$ are i.i.d. standard Gaussian $N(0, 1)$ and $c_J$ are deterministic constants determined by the scalar inner product $\langle , \rangle$. Note that covariance kernel of $f(x, y)$ is given by

$$K_n(\vec{s}, \vec{t}) := \mathbb{E}[f(s_1, s_2)f(t_1, t_2)] \text{ for } \vec{s}, \vec{t} \in \mathbb{R}^2.$$  \hfill (1)

Let us denote the Gaussian vectors by $W = f$ and $T = xf_x + yf_y$. We also let

$$\Sigma := \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}$$

be the covariance matrix of the Gaussian vector $(W, T)$ where

$\alpha = \text{Cov}(W, W), \beta = \text{Cov}(W, T) \text{ and } \gamma = \text{Cov}(T, T)$. 

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Random Real Algebraic Geometry
Let $f(x, y) = 0$ be a Gaussian random real algebraic curve. Assume that
\[
\frac{1}{x^2 + y^2} \left( \frac{\sqrt{\det \Sigma}}{\alpha} + \frac{|\beta|}{\alpha^3} \right)
\]
is integrable near the origin in $\mathbb{R}^2$. Then
\[
\mathbb{E}[\ell_p] = \frac{1}{2\pi} \int\int_{\mathbb{R}^2} \mathbb{E}[|xf_x + yf_y| \mid f = 0] \frac{dxdy}{x^2 + y^2} + C_n
\]
\[
= \frac{1}{2\pi^2} \int\int_{\mathbb{R}^2} \frac{\sqrt{\det \Sigma}}{\alpha} \frac{dxdy}{x^2 + y^2} + C_n.
\]
where $C_n$ is either 0 or $2\pi$ depending on degree $n$ is even or odd.
We introduce the differential operators

\[ D_1 := \frac{1}{2} s_1 \frac{\partial}{\partial s_1} \quad \text{and} \quad D_2 := \frac{1}{2} t_2 \frac{\partial}{\partial t_2} \]

on \( \mathbb{R}^2 \times \mathbb{R}^2 \). Note that

\[
\frac{\sqrt{\det \Sigma}}{\alpha} = (D_1 + D_2)^2 \log K_n(\vec{s}, \vec{t})|_{\vec{s} = \vec{t} = (x, y)}.
\] (4)

In particular,

\[
\mathbb{E}[\ell_p] = \frac{1}{2\pi^2} \int \int_{\mathbb{R}^2} \sqrt{(D_1 + D_2)^2 \log K_n(\vec{s}, \vec{t})}|_{\vec{s} = \vec{t} = (x, y)} \frac{dxdy}{x^2 + y^2}
\] (5)
Now, we apply our formula to Complex Fubini-Study Ensemble. We remark that by invariance property of this ensemble the expected depth of a random real algebraic curve is independent of the choice of the point $p \in \mathbb{RP}^2$. We have the following:

**Corollary**

*We let $W(2, n)$ be endowed with the complex Fubini-Study probability measure. For every $p \in \mathbb{RP}^2$*

\[
\mathbb{E}[\ell_p] = \frac{\sqrt{n}}{2}.
\]

**Proof.**

It follows from binomial formula that $K_n(x, y) = (1 + x^2 + y^2)^n$ and applying the equation (5) above gives the result.
In a more recent series of papers, GW studied $\mathbb{E}[N(V_{f_n})]$ the expected number of connected components and they proved that

\begin{theorem}[Gayet-Welschinger, 2012-2014]
We consider $W(m, n)$ endowed with the complex Fubini-Study probability measure. Then there exists $0 < C_1 < C_2$ such that

$$C_1 n^{\frac{m}{2}} \leq \mathbb{E}[N(V_{f_n})] \leq C_2 n^{\frac{m}{2}} \text{ as } n \to \infty.$$ 

\end{theorem}

In dimension one, their results give sharp bound on expected number of real zeros. In the case of real algebraic plane curves (i.e. $m = 2$) the above result gives the asymptotics for the expected number of ovals (since there is either one pseudoline or none, depending on the parity of the degree, it does not affect the asymptotics).
Other models also have been considered in the literature. For instance, Nazarov and Sodin (2009, Amer. J. Math.) studied the number of nodal domains of random spherical harmonics.

Building upon ideas of Nazarov and Sodin recently Lerario and Lundberg proved that in the real Fubini-Study ensemble the order of number of connected components is $d^2$. More, precisely,

**Theorem (Lerario and Lundberg, 2014 IMRN)**

We consider $W(m, n)$ endowed with the real Fubini-Study probability measure. Then

$$\mathbb{E}[N(V_{f_n})] = O(n^m).$$
Next, we will consider the asymptotic growth of Expected projective volume $\mathbb{E}[\text{Vol}(V_f)]$. Using the notation of the previous section for $f \in W(m, n)$ we assume that the equation of $V_f$ in the affine chart $\{Z \neq 0\}$ is given by $f(x, y)$.

For a linear map $L : \mathbb{R}^m \to \mathbb{R}^m$ we denote by $|\det \perp L| := \sqrt{\det(LL^*)}$ where $L^*$ denotes the adjoint of $L$. Then Kac-Rice formula has a generalization to the current setting:

**Theorem (Kac-Rice)**

\[
\mathbb{E}[\text{Vol}(V_f)] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^m} \mathbb{E}[|\det \perp f'(x)||f(x) = 0]] \frac{1}{\sqrt{K_n(x, x)}} \, dx
\]
### Expected growth of Volume

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THANK YOU!!!!