A New Representation for Quartic Curves and Complete Sets of Geometric Invariants

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Abstract

Many free-form object boundaries can be modeled by quartics with bounded zero sets. The fact that any non-degenerate closed-bounded algebraic curve of even degree \( n = 2p \) can be expressed as the product of \( p \) conics, which are real ellipses, plus a remaining polynomial of degree \( n - 2 \) [12] can be utilized to express a non-degenerate quartic as the product of two leading ellipses plus a third conic which might be either a closed curve (an ellipse) or an open curve (a hyperbola). However, it can be shown that the leading ellipses can be modified with appropriate constants by constraining the third conic to be a circle, thus implying a 2-ellipse and 1-circle; i.e. an elliptical-circular \((E^2C)\) representation of the quartic. The use of such representations is to simplify the analysis of quartics by exploiting the well-known properties of conics and to develop a set of functionally independent geometric invariants for recognition purposes. Also, it is shown that the underlying Euclidean transformation between two configurations of the same quartic can be determined using the centers of the three conics.

KEY WORDS: Algebraic Curves, Geometric Invariants, Canonical Curves, Object Identification/Recognition, Intrinsic Coordinate Systems

1 Introduction

Many free-form objects represented by their boundary data points can be modeled by quartic algebraic curves with bounded zero sets. Here, we assume that the set of data points represents the boundary of a planar curve or the planar silhouette of a 3D surface. Moreover, we will employ a model-based approach, where we begin by fitting an algebraic curve to the data points. This approach has two important advantages over direct data point methods, namely data point correspondences are unnecessary, and object identification is less sensitive to both noise and missing data.

Conics are second degree algebraic curves which are used in many model-based applications. Ellipses, hyperbolas and parabolas are non-degenerate conics, whereas two parallel or intersecting lines (real or imaginary) are examples of degenerate conics. It is relatively easy to construct canonical frames for conics using their geometric centers and the eigenvectors of the conic matrix of their second degree terms.
Taubin generalized the idea of canonical frames for higher degree curves by introducing intrinsic coordinate systems for such curves\cite{8}. In another related work\cite{7} quartics are represented using the procedure outlined in\cite{11} and a complete set of geometric invariants are derived. The center of the third conic is constrained to be the mid-point of the line segment joining the centers of the two leading conics.

In this paper, we show that the decomposition procedure detailed in \cite{12} for any closed-bounded curve can be used to represent closed-bounded quartic curves as the product of two ellipses plus a circle. The constraint that makes the third conic a circle is linear. Some nice features about this representation are that it immediately yields a complete set of independent scalar invariants that are geometric quantities, and it implies the underlying Euclidean transformation. We restrict our attention to quartics since they are relatively simple and do not require an excessive number of coefficients.

2 Elliptical-Circular ($E^2C$) Representation of Quartic Curves

Quartics are algebraic curves of degree $n = 4$ that are defined in the Cartesian $\{x, y\}$-plane by the implicit polynomial (IP) equation

$$f_4(x, y) = a_{40}x^4 + a_{31}x^3y + a_{22}x^2y^2 + a_{13}xy^3 + a_{04}y^4 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00} = 0$$

(2.1)

Since (2.1) can be multiplied by any non-zero scalar without changing the zero set, we often will find it convenient to deal with monic quartics, which are defined by the condition that $a_{40} = 1$.

Many different two-dimensional free-form objects can be modeled by quartics\cite{1, 2}, as illustrated in Figure 1. Moreover, algebraic invariants have been defined and used to identify such objects under various transformations\cite{3, 4, 8}. In \cite{12}, it was formally established that any closed-bounded non-degenerate (monic) curve $f_n(x, y) = 0$ of degree $n = 2p \geq 4$ can be uniquely represented as

$$f_n(x, y) = \prod_{k=1}^{p} C_{k0}(x, y) + f_{n-2}(x, y) = 0$$

(2.2)

where $f_{n-2}(x, y)$ is a polynomial of degree $n - 2$, and each conic factor

$$C_{k0} = x^2 + q_kxy + r_ky^2 + s_kx + t_ky$$

(2.3)
The subscript $k_0$ in $C_{k_0}(x,y)$ is used to emphasize the fact that its constant term is zero. If $4r_k \neq q_k^2,$ each conic factor of (2.2) will have a unique (conic factor) center $c_k = \{x_{kc}, y_{kc}\}$ defined by the simultaneous solution of the linear equations

$$\frac{\partial C_{k_0}(x,y)}{\partial x} = 2x + q_k y + s_k = 0 \quad \text{and} \quad \frac{\partial C_{k_0}(x,y)}{\partial y} = q_k x + 2r_k y + t_k = 0 \quad (2.4)$$

Note that the addition of a non-zero scalar term $v_k$ to $C_{k_0}(x,y)$ does not change the position of its center. Also, $C_{k_0}(x,y) + v_k \overset{def}{=} C_k(x,y) = 0$ will remain an ellipse, although it may be imaginary[14].

**Theorem:** Any closed-bounded non-degenerate quartic curve defined by (2.2) can be uniquely represented as the product of two ellipses plus a circle.

**Proof:** For closed-bounded quartics, (2.2) implies that

$$f_4(x,y) = \frac{(x^2 + q_1 xy + r_1 y^2 + s_1 x + t_1 y)(x^2 + q_2 xy + r_2 y^2 + s_2 x + t_2 y) + f_2(x,y) = 0 \quad (2.5)$$

for some $f_2(x,y) = b_{20} x^2 + b_{11} xy + b_{02} y^2 + b_{10} x + b_{01} y + b_{00}.$ The addition of $v_1$ and $v_2$ to $C_{10}(x,y)$ and $C_{20}(x,y)$ in (2.5) then implies that

$$\left[\begin{array}{c} C_{10}(x,y) \\
C_{20}(x,y) \end{array}\right] \left[\begin{array}{c} v_1 \\
v_2 \end{array}\right] = \left[\begin{array}{c} C_{10}(x,y)C_{20}(x,y) + v_1 C_{20}(x,y) + v_2 C_{10}(x,y) + v_1 v_2, \end{array}\right] \quad (2.6)$$

or that

$$f_4(x,y) = C_1(x,y)C_2(x,y) + f_2(x,y) - v_1 C_{20}(x,y) - v_2 C_{10}(x,y) - v_1 v_2,$$

$$C_3(x,y)$$

$$3$$
where
\[ \tilde{C}_3(x, y) = (b_{20} - v_1 - v_2)x^2 + (b_{11} - v_1q_2 - v_2q_1)xy + (b_{02} - v_1r_2 - v_2r_1)y^2 
+ (b_{10} - v_1s_2 - v_2s_1)x + (b_{01} - v_1t_2 - v_2t_1)y + b_{00} - v_1v_2 \]

We now define \( v_1 \) and \( v_2 \) as solutions to the matrix/vector equation:
\[
\begin{bmatrix} 1 - r_2 & 1 - r_1 \\ q_2 & q_1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} b_{20} - b_{02} \\ b_{11} \end{bmatrix},
\]
which implies that \( b_{11} - v_1q_2 - v_2q_1 = 0 \) and that \( b_{20} - v_1 - v_2 = b_{02} - v_1r_2 - v_2r_1 \) def \( \kappa \). It follows that \( \tilde{C}_3(x, y) = \kappa C_3(x, y) \), with \( C_3(x, y) = 0 \) a (monic) circle defined by
\[
C_3(x, y) = x^2 + y^2 + 2Ax + 2By + C = 0,
\]
with center at \( (x_c, y_c) = (-A, -B) \) and with radius \( r = (A^2 + B^2 - C)^{1/2} \); i.e.
\[
f_4(x, y) = C_1(x, y)C_2(x, y) + \kappa C_3(x, y) = 0 \quad \square
\]

Figures 2 and 3 display our unique \( E^2C \) representation for two different quartic curves. In both cases depicted, the ellipses and the circles are real, although this is not true in general. Figure 3 also depicts the centers of the three conics and the geometric base triangle they define. This base triangle can be used to determine the unique planar configuration (position and orientation) of the quartic curve.

Under Euclidean transformations, conics map to identical conics that are rotated by an angle \( \theta \) and translated in the plane. Therefore, the two ellipses and the circle defined by (2.8) will map to two identical ellipses and an identical circle. This observation serves to illustrate a complete geometric set of Euclidean invariants for quartic curves, in the next section, which can be used for object recognition.

It is of interest to note that our representation is not unique under affine transformations, since circles do not generally map to corresponding circles under affine transformations. However, affine equivalent boundary data sets that imply quartic IP representations can be normalized[12] in order to apply the results presented here. Furthermore, in the affine case, we have developed a unique representation for IP curves of any degree \( n \) in terms of conic-line products[9, 10, 13]. These representations imply many different absolute algebraic invariants and numerous corresponding related-points that also can be used for object identification and alignment.
3 A Complete Set of Geometric Invariants

Any monic conic equation \( C(x, y) = x^2 + 2hxy + by^2 + 2cx + 2dy + e = 0 \) can be written as

\[
C(x, y) = \begin{bmatrix} x & y & 1 \end{bmatrix} X^T Q X \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0 \tag{3.9}
\]

where \( Q \) is a \( 3 \times 3 \) symmetric matrix. In light of (2.8) and (3.9), a (monic) \( E^2C \) quartic curve can be written as

\[
f_4(x, y) = X^T Q_1 X X^T Q_2 X + \kappa X^T Q_3 X = 0 \tag{3.10}
\]

C1(x, y) C2(x, y) C3(x, y)

If such a curve undergoes a Euclidean transformation \( E \) defined by

\[
\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \end{pmatrix} \quad \iff \quad \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta & t_x \\ \sin \theta & \cos \theta & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ 1 \end{pmatrix}, \tag{3.11}
\]

\[
\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \quad \iff \quad \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta & t_x \\ \sin \theta & \cos \theta & t_y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ 1 \end{pmatrix}, \tag{3.11}
\]
the transformed $E^2C$ quartic curve will be defined by

$$f_3(\bar{x}, \bar{y}) = \underbrace{\bar{X}^T E^T Q_1 E \bar{X}}_{\bar{C}_1(\bar{x}, \bar{y})} + \underbrace{\bar{X}^T E^T Q_2 E \bar{X}}_{\bar{C}_2(\bar{x}, \bar{y})} + \underbrace{\kappa \bar{X}^T E^T Q_3 E \bar{X}}_{\bar{C}_3(\bar{x}, \bar{y})} = 0$$  \hspace{1cm} (3.12)

Note that the scalar $\kappa$ remains invariant under $E$, so that $\kappa$ is a Euclidean invariant.

Now consider a hypothetical quartic curve defined by (2.8) and (3.10), not explicitly shown, that has been (uniquely) represented as the product of two ellipses and a circle, all of which are real, as shown in Figure 4. Since a quartic curve has 14 independent coefficients, and a Euclidean transformation is defined by one rotation and two translations, the invariant counting argument[5] implies that there are $14 - 3 = 11$ Euclidean invariants. As noted above, the scalar $\kappa$ represents one of these invariants.

The remaining 10 Euclidean invariants are geometric, and immediately obvious in Figure 4, namely the 4 major/minor half-axes lengths, $a_1, b_1, a_2, b_2$ of the two ellipses, the radius $R$ of the circle, the 3 side lengths $d_1, d_2$ and $d_3$ that define the geometric base triangle, and the two angles $\phi_1$ and $\phi_2$ that define the orientation of the ellipses relative to the circle$^1$. The center of the circle and the directions of $d_1$ or $d_2$ will define intrinsic coordinate systems analogous to those defined in [7] and [8].

Note that one can also define two new invariants as the ratio of the major and minor axes of the ellipses, namely $a_1/b_1$ and $a_2/b_2$. These two ratio invariants are used in our experiments to show the relative positions of various objects in a two-dimensional invariant space.

We next consider the case of imaginary ellipses. In particular, it is well known[14] that a Euclidean transformation can be used to translate the center of a real ellipse to the origin of its Cartesian coordinate system and to rotate its axes so that they align with the $x$ and $y$ axes. The resulting (real) canonical ellipse will be defined by the equation:

$$C(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

with $a$ and $b$ the half-axes lengths.

For imaginary ellipses, an analogous transformation will imply the (imaginary) canonical

$^1$It might be noted that Figure 4 implies many alternative and additional geometric invariants defined by distances and angles. However, the functionally independent set of 10 geometric invariants noted above will imply all such alternative invariants.
ellipse

\[ C(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0, \]

with \( a_i \) and \( b_i \) the “imaginary” half-axes lengths, which clearly have a real distance interpretation. Moreover, imaginary ellipses also have real centers defined by (2.4) and real lines (through their centers) that define the directions of their major and minor axes[14].

In light of these observations, it follows that imaginary ellipses and circles imply real geometric invariants under Euclidean transformations that are completely analogous to those depicted for real ellipses and circles in Figure 4.

We might also note that an unbounded quartic curve that has only two asymptotes can be uniquely represented as the product of an ellipse and a hyperbola plus a circle. Such an \( EHC \) representation also implies a complete set of geometric invariants, analogous to those depicted in Figure 4, if the half-axes lengths of one of the ellipses are replaced by the transverse and conjugate half-axes lengths of the hyperbola.
4 Alignment

The two elliptical centers and the circular centers of two Euclidean equivalent quartics directly imply the Euclidean transformation matrix which relates them, since such centers are related-points[12] which map to one another under Euclidean transformations. In particular, three corresponding centers $O_i = (x_i, y_i)$ and $\bar{O}_i = (\bar{x}_i, \bar{y}_i)$, for $i = 1, 2$ and 3 of two equivalent quartics, will satisfy the relation:

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{bmatrix} = E \begin{bmatrix} \bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\ \bar{y}_1 & \bar{y}_2 & \bar{y}_3 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow E = A\bar{A}^{-1}$$

To determine the correct correspondence among the three centers, each curve can be evaluated at these conic centers, since the resulting function values must be proportional to one other. For example, the relations

$$z_1 = f_4(O_1) < z_2 = f_4(O_2) < z_3 = f_4(O_3)$$

and

$$\bar{z}_1 = \bar{f}_4(\bar{O}_1) < \bar{z}_2 = \bar{f}_4(\bar{O}_2) < \bar{z}_3 = \bar{f}_4(\bar{O}_3)$$

will establish a correct correspondence among the three conic centers.

We finally remark that the unique representation of all IP curves outlined in [9] and [13] shows that when $n = 2p$, $f_n(x, y)$ can be expressed as the sum $f_n(x, y) = F(x, y) + f_4(x, y)$, where $F(x, y)$ and $f_4(x, y)$ will map, respectively, to an equivalent $\bar{F}_n(\bar{x}, \bar{y})$ and an equivalent $\bar{f}_4(\bar{x}, \bar{y})$ under Euclidean transformations. If such an $f_4(x, y) = 0$ is either a closed-bounded curve or an unbounded curve with two asymptotes, our unique representations will imply geometric invariants and related-points for (the lower degree terms of) $f_n(x, y)$ that can be used to identify and align $f_n(x, y) = 0$.

5 Affine Equivalent Quartics

In [12], we introduce affine canonical matrices, using the centers of the conics and the critical points of the curves, which in turn imply canonical curves. We then prove that affine equivalent curves have the same canonical curve. Based on the results presented in earlier sections, one
can employ our elliptical-circular decomposition, \( E^2C \), for quartic canonical curves to obtain geometric invariants instead of directly comparing the coefficients of the canonical curves.

More precisely, the (monic) canonical curve \( f_4^e(x, y) = 0 \) of \( f_4(x, y) = 0 \) will be defined by the affine canonical transformation matrix

\[
\hat{A} \overset{\text{def}}{=} \begin{bmatrix}
 x_1 - x_3 & x_2 - x_3 & x_3 \\
 y_1 - y_3 & y_2 - y_3 & y_3 \\
 0 & 0 & 1
\end{bmatrix} \quad \Longrightarrow \quad f_4(x, y) = 0 \quad \overset{\hat{A}}{\rightarrow} \quad \hat{s} f_4^e(x, y) = 0, \quad (5.13)
\]

and the (monic) canonical curve \( \tilde{f}_4^e(x, y) = 0 \) of \( \tilde{f}_4(x, y) = 0 \) will be defined by the affine canonical transformation matrix

\[
\hat{A} \overset{\text{def}}{=} \begin{bmatrix}
 \bar{x}_1 - \bar{x}_3 & \bar{x}_2 - \bar{x}_3 & \bar{x}_3 \\
 \bar{y}_1 - \bar{y}_3 & \bar{y}_2 - \bar{y}_3 & \bar{y}_3 \\
 0 & 0 & 1
\end{bmatrix} \quad \Longrightarrow \quad \tilde{f}_4(x, y) = 0 \quad \overset{\hat{A}}{\rightarrow} \quad \hat{s} \tilde{f}_4^e(x, y) = 0 \quad (5.14)
\]

where \( O_i = (x_i, y_i) \) and \( \bar{O}_i = (\bar{x}_i, \bar{y}_i) \), for \( i = 1, 2 \) and 3 are three corresponding related-points of the affine equivalent curves.

If these two monic canonical curves, \( f_4^e(x, y) = 0 \) and \( \tilde{f}_4^e(x, y) = 0 \), are the same, their \( E^2C \) representations will naturally imply the same geometric invariants. Hence, we can associate a canonical curve with each quartic in a database off-line and define a set of geometric invariants for these canonical curves. When a new object is given, we can compute its invariants and compare them with the stored invariants to see if the new object belongs to the database.

### 6 Experiments

We now will present some experimental results which illustrate our procedures. In these experiments, it is assumed that objects have already been segmented from their background and their boundaries have been extracted. Thus, we start by fitting quartics to object boundaries represented by a finite number of points. The data sets in Figure 5 (400-600 point sets which appear to be continuous) depict six different test objects used in our experiments. The smoother curves are our superimposed 4th degree IP curves obtained using the fitting algorithm outlined in [1]. Note that most of these objects are too complicated to be precisely by quartics. Nevertheless, quartic approximations prove to be most useful in identifying these objects, as we will now show.

To test the robustness and the discrimination power of our invariants, in the first set of experiments data sets were sub-sampled and subjected to gaussian noise of variance \( \sigma^2 = 0.05 \).
Figure 5: Test objects used in experiments and superimposed quartics, a: airplane, b: butterfly, c: guitar, d: tree, e: mig29, f: boot

We then applied 100 random Euclidean transformations to these noisy data sets which were subsequently modeled by quartics.

<table>
<thead>
<tr>
<th></th>
<th>$a_1/b_1$</th>
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<td>0.8319</td>
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Table 1: Percentage relative error of invariants under gaussian noise with $\sigma^2 = 0.05$

The percentage relative error of 5 Euclidean invariants, namely $a_1/b_1,a_2/b_2,d_1,d_2$ and $d_3$ under gaussian noise of variance $\sigma^2 = 0.05$ are computed and tabulated in Table 1. From this table it is clear that the two ratio invariants, $a_1/b_1$ and $a_2/b_2$, are more robust than the invariant distances, $d_1,d_2$ and $d_3$. Note also that the invariant distance $d_3$ is more robust than the invariant distances $d_1$ and $d_2$. The reason for this is that $d_3$ is the distance between the centers of the leading ellipses, whereas $d_1$ and $d_2$ are the distances between the centers of the leading ellipses and the center of the circle. We know that the centers of the leading ellipses are independent of their constant terms and thus they are determined from the coefficients of
the third and fourth degree terms of the quartic. On the other hand, the center of the circle is determined from the coefficients of the second degree terms. Under noisy conditions, it was experimentally observed that the fitting procedure used here yields curves whose higher degree terms are more robust than their lower degree terms, which implies that the higher degree terms are less sensitive to noise since they are more dominant in the shape representation. These observations motivate the use of two ratio invariants to represent the relative location of objects in 2D invariant space and the use of invariant distance $d_3$ when two objects overlap.

<table>
<thead>
<tr>
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Table 2: Percentage relative error of invariants under 5% missing data

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</table>

Table 3: Percentage relative error of invariants under 10% missing data

Figure 6 depicts the location of the ratio invariants of the test objects in 2D invariant space. Note that all of the objects are well separated in this invariant space, except for the boot and the tree. Clearly, we could use additional invariants to discriminate the boot from the tree. Figure 7 depicts the 2D invariant space when the invariant distance $d_3$ between the centers of the two leading ellipses, $C_1(x,y) = 0$ and $C_2(x,y) = 0$, is used together with the ratio invariant $a_1/b_1$ under similar experimental conditions. Note that the boot and the tree are now separated from each other.

In the final set of experiments, the boundary data sets were transformed with random Euclidean transformations and certain subsets of the data points were chopped at different boundary locations to test the robustness of our invariants relative to missing data. The percentage
Figure 6: The test objects in 2D invariant space using the ratio invariants, $a_1/b_1$ and $a_2/b_2$

Figure 7: The boot and the tree in 2D invariant space using the invariants $a_1/b_1$ and $d3$
relative error of our invariants under 5% and 10% missing data are computed and shown in Tables 2 and 3, respectively. As in the noisy case, the two ratio invariants are more robust when compared to the remaining invariant distances. Similarly, $d_3$ is the most robust invariant distance. In these missing data experiments, the robustness of invariants is a direct consequence of the interpolation properties of the implicit polynomial curves through missing data.

7 Concluding Remarks

We have shown that closed bounded quartic curves can be represented as the product of two ellipses plus a circle, which can be real or imaginary. Such representations imply three identical conics under Euclidean transformations, and the centers of the three sets of equivalent conics directly imply the underlying transformation. A new set of geometric invariants were defined using this representation, and extensions to unbounded and higher degree curves were discussed. The robustness and the discrimination capabilities of these invariants were tested with noisy and missing data. In particular, the ratio invariants were found to be quite robust when compared to invariant geometric distances. Finally, our $E^2C$ and $EHC$ planar representations suggest the possibility of analogous representations in the three dimensional case.

References


