ABSTRACT

The problem of gear noise in vehicles has been intensively studied in the past; however, recently the interest about this problem grew because of great restrictions in the laws regarding noise level and the increase of international competition. One of the most important vibration and noise sources is transmission error that excites the gearbox as a dynamic system, the gearbox surfaces, and connected components; the external box radiates noise. However, the current understanding of gear vibration remains incomplete, even though there is general agreement about the nature of the phenomenon. Vibrations are due to several sources: torsion resonance, impulsive or cyclic fluctuations in drive torque, gear mesh transmission error, local component vibration responses and fluctuations in the output torque demand.

The concept of a vibrating system made of two gears is generally modeled through two wheels linked by the teeth mesh stiffness. In its simplest form, this model can simulate the classical linear resonance, i.e. the resonant frequency of the system. However, more complex phenomena such as parametric instabilities can be an important source of noise.

In the present paper vibration problems in the gears of an industrial vehicle are investigated through the use of perturbation technique. A suitable software has been developed to generate the gear profiles in order to evaluate global mesh stiffness using finite element analysis.

INTRODUCTION

In the analysis of gear dynamics several engineering topics are generally considered: geometric profile/surface generation, kinematics; gear tooth, shaft and support elasticity; linear and nonlinear dynamical interactions.

In the past several approaches have been proposed for gear modeling. An interesting literature overview can be found in [1], where the mathematical models used in gear dynamics were classified by considering: the evaluation of simple dynamic factor; tooth compliance; gear dynamics; geared motor dynamics; and torsion vibration.

Helical gears were studied by Andersson and Vedmar [2], teeth impact problems were investigated by Yan and Sun [3], the flexibility of shafts using a multi-degrees of freedom model was analyzed by Lin et al. [4,5]. Cai and Hayashi [6] proposed a linear approximation for a pair of spur gears and compared the analytical solution with the numerically calculated result by the nonlinear equation; they found that linear models are able to capture the main dynamic phenomena. Parker et al. [7] studied the nonlinear dynamic response of a spur gear pair using a semi-analytical approach and two single degrees of freedom models. Amabili and Rivola [8] obtained a continuous closed form solution for any rotational speed and computed the transition curves, stable and unstable regions, by means of the Hill infinite determinant.
The evaluation of the model parameters as mesh stiffness, mesh damping and transmission error is another important aspect of gear modeling: finite element analysis, analytical models and experiments are the commonly used approaches. Ming-Haung Tsai and Ying-Chien Tsai [9] compared FEM result with the Houser and Tavakoli’s [10] analytical technique for a plastic spur gear pair. Howards et al. [11] used FEM to ascertain the dynamical effect of different types of gear train damage.

One of the main problems in using FEM is the geometry generation; in particular, a good tooth profile and tooth root fillet generation is quite important to obtain reliable results. Such generation can be obtained from analytical or numerical approaches. Existence conditions for envelope curves were intensively studied by Zalgaller and Litvin [12]. Litvin et al. [13] focused their attention on gear profiles, and proposed methods for finding spurious envelope solutions.

In the present paper, software has been developed to build the teeth profile. The software is able to generate a very large classes of 2D gears, starting from any rack shape. Using this technique the tooth fillet profile, including undercutting, can be modeled. Then a single degree of freedom system has been developed for the dynamic modeling. The stiffness of the gears is calculated using commercial finite element software (FEM) for different meshing positions of a gear pair, using contact elements for the teeth interaction. This approach allows one to accurately evaluate the variable global stiffness, which depends on the gear position, materials and teeth geometry. The variable stiffness gives rise to a parametric excitation, which is periodic; therefore it gives rise to a Mathieu type instability. A Fourier series of the variable stiffness is obtained from the numerical data evaluated with the FEM model. The dynamic model is analyzed by means of perturbation techniques [14].

**MESHING EQUATIONS**

Let us consider the profile generation by means of an envelope of a rack profile \( \sigma_1 \), see fig. 1. Three references systems are introduced:

- \( S_f(o_x y_f) \) fixed reference;
- \( S_1(o_1 x_1 y_1) \) moving respect to \( S_f \);
- \( S_2(o_2 x_2 y_2) \) moving respect to \( S_f \);

where \( \sigma_1 \) is fixed in \( S_1 \) and \( \sigma_2 \) is fixed in \( S_2 \). \( \theta \) is local variable that identify a point on the curve \( \sigma_1 \) and \( \phi \) is local variable that identify a point on the curve \( \sigma_2 \). The parameter \( \phi \) depends on \( \phi \); \( P_1 \) and \( P_2 \) are the same point \( P \) represented in the reference systems \( S_1 \) and \( S_2 \).

![Fig. 1: Reference systems](image)

Let us consider a regular plane curve \( \sigma_1 \) describing the rack tooth profile. The tooth profile \( (\sigma_2) \) is described by the meshing equation:

\[
f(\theta, \phi) = \tilde{n}_1(\theta) \cdot \tilde{v}_c(\theta, \phi) = 0 \tag{1}\]

this equation states that the sliding velocity \( \tilde{v}_c \), of \( \sigma_2 \) with respect to \( \sigma_1 \), has to be normal to the enveloping curve \( \sigma_1 \).

Moreover one can express the normal vector to curve \( \sigma_1 \) as:

\[
\tilde{n}_1(\theta) = \kappa_1 \times \frac{\partial}{\partial \theta} \tilde{O}_1 \tilde{P}_1(\theta) \tag{2}\]

Using homogeneous coordinates, \( P_1 \) on \( \sigma_1 \) is described by vector \( \tilde{O}_1 \tilde{P}_1 \). The following regularity conditions of the curve are imposed:

\[
\tilde{O}_1 \tilde{P}(\theta) \in C^1, \quad \frac{\partial \tilde{O}_1 \tilde{P}_1}{\partial \theta}(\theta) \neq 0 \tag{3}\]

Assuming that \( S_1 \) is fixed and \( S_2 \) is moving with respect to \( S_1 \), the sliding velocity of \( P \) in \( S_1 \) becomes:

\[
\tilde{v}_c(P) = \frac{d}{dt} \tilde{O}_1 \tilde{P} + \tilde{\omega} \times \tilde{O}_1 \tilde{P} = \frac{d}{dt} \tilde{O}_1 \tilde{O}_2 + \tilde{\omega} \times (\tilde{O}_1 \tilde{O}_2 + \tilde{O}_1 \tilde{P}) \tag{4}\]

In order to describe the vector \( \tilde{O}_1 \tilde{P}_1(\theta) \) in the reference \( S_2 \) a transformation matrix \( M_{21} \) must be considered:

\[
\tilde{O}_2 \tilde{P}_2(\theta, \phi) = M_{21}(\phi) \cdot \tilde{O}_1 \tilde{P}_1(\theta) \tag{5}\]

\( M_{21} \) transforms reference system 1 to 2 by expressing the motion of \( S_1 \) and \( S_2 \) with respect to \( S_f \) using the Lagrange
parameter $\phi$. Equation (5) represents a family of curves, which describe the trajectories of the points of $\sigma_1$ while $\theta$ is varying during the motion respect to $S_2$. Equation (1) allows to reduce the family of curves to a unique curve $\sigma_2$, which is the envelope of $\sigma_1$. Equation (1) and (5) are sufficient to determinate the tooth profile curve $\sigma_2$ as function of $\phi$.

According to Litvin, et al. [13], the following sufficient conditions for the existence and regularity of the curve must be respected:

1) The family of curves $\sigma_2$ must have $C^2$ regularity for both $\theta$ and $\phi$ parameters:

$$\frac{\partial}{\partial \theta} O_2 P_2(\theta, \phi) \neq 0$$

2) $\sigma_1$ must be regular:

$$\frac{\partial}{\partial \theta} O_1 P_1(\theta_0) \neq 0$$

3) If a generic point $(\theta_0, \phi_0)$ verifies eq. (1), then it cannot be a singular point and must verify the following equation:

$$\frac{\partial}{\partial \theta} O_1 P_1(\theta) + \nu_1(\theta_0, \phi_0) \neq 0$$

where $\theta = \theta(t)$ provides the contact point on $\sigma_1$ while $\theta$ is varying.

APPLICATION OF THE ANALYTICAL ENVELOPING METHOD

Sliding velocity

From fig. 1 one obtains that:

$$O_1 O_2 = R \phi \overline{v}_1 - R \overline{j}_1 = \left[ R \phi \quad -R \quad 0 \quad 0 \right]$$

and

$$\overline{\omega} = \left[ 0 \quad 0 \quad \omega \quad 0 \right]^T, \quad \omega = \frac{d}{dt} \phi(t)$$

the sliding velocity given by eq. (4) becomes:

$$\overline{v}_1(P) = R \omega \overline{i}_1 + \begin{bmatrix} 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & \omega \end{bmatrix}$$

Roto-translation matrix

The rotation matrix is given by:

$$R(\eta) = \begin{bmatrix} \cos(\eta) & -\sin(\eta) & 0 & 0 \\ \sin(\eta) & \cos(\eta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $\eta$ is a generic rotation angle.

The translation of point P can be described by the translation matrix:

$$P + \overline{t} = \begin{bmatrix} P_x + t_x \\ P_y + t_y \\ P_z + t_z \end{bmatrix} = T \overline{P}$$

where $\overline{t}$ is a vector indicating the translation.

Finally, defining $\eta = \phi - \pi/2$ and $\overline{t} = O_1 O_2$ the complete roto-translation matrix becomes:

$$M_{\eta}(\phi) = \begin{bmatrix} -\sin(\phi) & \cos(\phi) & 0 & R(\phi \sin(\phi) + \cos(\phi)) \\ -\cos(\phi) & -\sin(\phi) & 0 & R(\phi \cos(\phi) - \sin(\phi)) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rack protuberance parameters

Figure 2 shows the rack and protuberance parameters according to UNI 8862/2 [15]:
\[
\begin{aligned}
\begin{cases}
x_i = x_c + r \cos \left( \frac{\theta_i}{r} \right) \\
y_i = y_c + r \sin \left( \frac{\theta_i}{r} \right)
\end{cases}
\end{aligned}
\tag{15}
\]
where:
\[
\chi = \frac{\pi m}{4}
\]
\[
x_c = \chi + \frac{P_m}{\cos(\alpha_s)} - (h m - r \sin(\alpha_s)) \tan(\alpha_s) - r \cos(\alpha_s)
\]
\[
y_c = -h m + v + r
\]
\[
x_{T2} = x_c + r \cos(\alpha_{\rho v}) + (h m - v - r + r \sin(\alpha_{\rho v})) \tan(\alpha_{\rho v})
\]
\[
x_{T3} = -\chi - v \tan(\alpha_s)
\]
\tag{18}
\]
are the coordinates of the C, T₂ and T₃ points indicated in fig. 2. The three curves, which constitute the whole rack profile, are defined in the following intervals:

\begin{align}
(1) \theta_1 & \in \left[ \frac{\pi}{2} r, \theta_2 = -\alpha_{\rho v}, \pi \right] \\
(2) \theta_2 & \in \left[ \frac{\pi}{2} r - \frac{h m - v - r + r \sin(\alpha_{\rho v})}{\cos(\alpha_s)} \tan(\alpha_{\rho v}) - \frac{h m - v - r + r \sin(\alpha_{\rho v})}{\cos(\alpha_s)} \tan(\alpha_{\rho v}) \right] \\
& \text{if } \alpha_{\rho v} \neq \alpha_s \\
(3) \theta_3 & \in \left[ \frac{\pi}{2} r - \frac{h m - v - r + r \sin(\alpha_{\rho v})}{\cos(\alpha_s)} \tan(\alpha_{\rho v}) - \frac{h m + v}{\cos(\alpha_s)} \tan(\alpha_{\rho v}) \right] \\
& \text{if } \alpha_{\rho v} = \alpha_s
\end{align}
\tag{19}
\]

**Involute profile generated from a rectilinear segment**

Using eq. (16) or eq. (17) yields
\[
\frac{\overline{O_i R_i}}{\theta \cos(\alpha)} = \begin{bmatrix}
x_i + \theta \sin(\alpha) \\
\theta \cos(\alpha) \\
0 \\
1
\end{bmatrix}
\tag{20}
\]
which can be inserted in (1). Therefore the equation of mesh becomes:
\[
\omega \left[ \theta + \left( x_T - R \phi \right) \sin(\alpha) \right] = 0
\tag{21}
\]

which gives
\[
\theta(\phi) = (R \phi - x_T) \sin(\alpha)
\tag{22}
\]

Substituting (22) into (5) one obtains
\[
\begin{align}
\begin{cases}
x(\phi) = R \cos(\phi) + (R \phi - x_T) \cos(\alpha) \sin(\alpha + \phi) \\
y(\phi) = -R \sin(\phi) + (R \phi - x_T) \cos(\alpha) \cos(\alpha + \phi)
\end{cases}
\end{align}
\tag{23}
\]

**Profile generated from rack tooth fillet**

Equation (15) yields
\[
\omega \left[ y_c \cos \left( \frac{\theta}{r} \right) + (R \phi - x_c) \sin \left( \frac{\theta}{r} \right) \right] = 0
\tag{24}
\]

Solving eq. (25) with respect to \( \theta \), four solutions can be found in the interval \([-\pi, \pi]\):
\[
\theta_{1,2,3,4} = \pm \arccos \left( \pm \frac{x_c - R \phi}{\sqrt{y_c^2 + (x_c - R \phi)^2}} \right)
\tag{26}
\]
according to eq.s (19) only the following solution is acceptable:
\[
\theta = -\arccos \left( \frac{|x_c - R \phi|}{\sqrt{y_c^2 + (x_c - R \phi)^2}} \right)
\tag{27}
\]

Substituting (27) into (5) yields
\[
\begin{align}
\begin{cases}
x(\phi) = A \cos(\phi) + B \sin(\phi) \\
y(\phi) = B \cos(\phi) + A \sin(\phi)
\end{cases}
\end{align}
\tag{28}
\]
Equations (23) and (28) represent the curves that generate the tooth profile from the rack cutter envelope.

**Limits values**

The three curves r1, r2 and r3 generate three profiles t1, t2 and t3, see fig. 3. The boundaries of the curves r1, r2 and r3, in terms of their parameters θ1, θ2 and θ3 are reflected on the boundaries of the envelopes t1, t2 and t3, in terms of φ1, φ2 and φ3.

\[
A = R + y_c - r \frac{y_c^2}{y_c^2 + (x_c - R \phi)^2}
\]  
\[
B = R \phi - x_c - \frac{r \sqrt{(x_c - R \phi)^2}}{y_c^2 + (x_c - R \phi)^2}
\]  

(29)

The first equation refers to the straight line segments r2 and r3 (see fig. 3) of the rack, while the second one refers to the tooth fillet. The limits φ can be obtained by substituting into (30) the values for φ found in (19).

**Evaluation of intersection points: a numerical technique.**

By analysing fig. 3 one can argue that some parts of the envelope curves can be spurious, because of undercutting, i.e. the theoretical curves cannot be practically built.

The actual intersection point A is given by:

\[
\begin{bmatrix}
  x_{t1}(\phi_1) = x_{t3}(\phi_3) \\
  y_{t1}(\phi_1) = y_{t3}(\phi_3)
\end{bmatrix}
\]  

(31)

in which \(x_{t1}\) and \(y_{t1}\) are the expressions (28) and \(x_{t3}\) and \(y_{t3}\) are the expressions (23) applied to segment r3 of the rack.

Using the Newton technique eq. (31) can be solved iteratively. However, this approach can give rise to numerical troubles. In order to circumvent such problems, eq. (31) can be replaced by considering the distance between two envelope curves; by minimizing such distance one can obtain the intersection point.

The squared distance between two points located on curves t1 and t3 can be expressed as:

\[
f(\phi_1, \phi_3) = \left[ x_{t1}(\phi_1) - x_{t3}(\phi_3) \right]^2 + \left[ y_{t1}(\phi_1) - y_{t3}(\phi_3) \right]^2
\]  

(32)

The stationary points of \(f\) correspond to the zeroes of the gradient:

\[
\nabla f(\phi_1, \phi_3) = 2 \begin{bmatrix}
  \frac{d}{d\phi_1} x_{t1}(\phi_1) \cdot \Delta x + \frac{d}{d\phi_1} y_{t1}(\phi_1) \cdot \Delta y \\
  \frac{d}{d\phi_3} x_{t3}(\phi_3) \cdot \Delta x - \frac{d}{d\phi_3} y_{t3}(\phi_3) \cdot \Delta y
\end{bmatrix},
\]

\[
\Delta x = x_{t1}(\phi_1) - x_{t3}(\phi_3)
\]

\[
\Delta y = y_{t1}(\phi_1) - y_{t3}(\phi_3)
\]  

(33)

The Newton technique is used to solve eq. (33), i.e. obtain the stationary point; a first trial value for the vector \(\phi^{(0)}\) is chosen. The stationary points can be minimum, maximum or saddle points. The purpose is to find the minimum; therefore, the following check is considered (k is the number of iterations):

\[
f(\phi_1^{(k+1)}, \phi_3^{(k+1)}) < \epsilon^2
\]  

(34)

For what concern \(\phi^{(0)}\), geometric considerations yield to choose:

\[
\phi^{(0)} = \begin{bmatrix}
  \phi_{12} \\
  \phi_{32}
\end{bmatrix}
\]  

(35)

**Undercutting**

High values of negative addendum modification give rise to undercutting at the tooth base, i.e. close to the intersection between t1 and t2, see fig. 3.
Figure 4 shows how the rack generates spurious envelope branches if it is too close to the wheel.

![Fig. 4: Generation of spurious enveloped curves by the rack profile](image)

By considering figs. 3 and 4 one finds that a singularity happens for:

$$\phi_{32} < \phi_u$$  \hspace{1cm} (36)

The inequality means that the arc profile $r_3$ begins to envelop spurious branch before reaching $\phi_u$, at this point the singularity is produced. $\phi_u$ can be evaluated using eq. (8); moreover, such singular value can be found graphically by observing fig. 4: when the gear angle is $\phi_u$ the rack has the following translation $x_T - R \tan(\alpha)$:

$$\phi_u = \frac{x_T}{R} - \tan(\alpha)$$  \hspace{1cm} (37)

In the case of undercutting at least two intersections occur: fig. 5 shows the case of undercutting between curves $t_1$ and $t_3$.

![Fig. 5: Intersections between curves $t_1$ and $t_3$](image)

Intersection B is known because it is generated by the limits of the two rack profiles $r_1$ and $r_3$, while the intersection A is evaluated with the previously mentioned numerical method, using $\phi_u$ instead of $\phi_{12}$ in eq. (35).

**Particular cases**

According to the values of the protuberance parameters, two particular cases can be obtained by the enveloping process.

![Fig. 6: Different protuberance values](image)

Figure 6 shows two situations, which are theoretical cases and are not included in standard protuberance parameters. In the first case (fig. 6(a)) the protuberance segment $r_2$ generates the small involute curve $t_2$ on the tooth in same way as $r_3$ forms $t_3$. So two intersection A and B must be calculate using numerical method. In the second case the highest values of protuberance $r_2$ annihilates the actual envelope of the segment $r_3$.

A criterion to understand when these two situations occur is to evaluate the intersection $\phi_p$ between curves $t_1$ and $t_3$. If the protuberance is such that the segment $r_2$ does not envelop the real tooth flank the following condition must be true:

$$\phi_p > \phi_{31}$$  \hspace{1cm} (38)

If the protuberance is such that the segment $r_3$ does not exist, the point correspondent to $\phi_{31}$ is above the tangent line to the circle tip. This implies:

$$\theta_{31} > \theta_{32}$$  \hspace{1cm} (39)

Equations (38) and (39) take into account the curves orientations as in fig. 6.
**2D-Gear generator program**

All the developed equations were implemented in a FORTRAN code. The code is able to generate an output script file for AutoCAD 2002 containing instructions to draw a complete 2D gear. The input file is inserted using dialog box and must contain the following rack and gear information:

- Module;
- Number of teeth;
- Pressure angle;
- Addendum modification;
- Rack fillet radius;
- Protuberance;
- Protuberance angle;

Figure 7 shows different teeth obtained from a rack without protuberance, with pressure angle 20°, module 1 mm and tooth fillet 0.4 mm, using different number of teeth and addendum modification.

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<td><img src="image14" alt="Image" /></td>
<td><img src="image15" alt="Image" /></td>
</tr>
</tbody>
</table>

**Fig. 7: Example of generated tooth from a rack without protuberance**

The code was used to generate the profile of two actual spur gears used in a tractor gearbox, whose parameters are listed below (fig. 8 shows a sample case of generation):

**Pinion:**
- Module = 3 mm
- Number of teeth = 28
- Pressure angle = 20°
- Addendum modification = 1.927 mm
- Rack fillet radius = 0.1 mm
- Protuberance = 0 mm
- Protuberance angle = 20°

**Gear:**
- Module = 3 mm
- Number of teeth = 43
- Pressure angle = 20°
- Addendum modification = 2.748 mm
- Rack fillet radius = 0.1 mm
- Protuberance = 0 mm
- Protuberance angle = 20°

**DYNAMIC MODEL**

Using a dynamic approach, spur gear can be modeled with two disks, see fig. 9, coupled by mesh stiffness and mesh damping; the radius of each disk is the base circle radius of the corresponding gear. In this model two assumptions are introduced: no tooth separation between contact teeth and no manufacturing errors on the gears.

**Fig. 8: Driver and driven gear profiles**

The rotational equations of motion of the pinion and gear are
\[ I_1 \ddot{\varphi}_1 + c(\dot{x}_1 - \dot{x}_2)R_1 + k(x_1 - x_2)R_1 = T_1 \]  
\[ (40) \]

\[ I_2 \ddot{\varphi}_2 - c(\dot{x}_1 - \dot{x}_2)R_2 - k(x_1 - x_2)R_2 = -T_2 \]  
\[ (41) \]

Where \( x_1 \) and \( x_2 \) are displacements along the line of action due to rotations \( \varphi_1 \) and \( \varphi_2 \) respectively.

Considering that \( x_i = \dot{\varphi}_i R_i, \) \( i=1,2 \) one can combine eq. (40) and eq. (41); the resulting differential equation for the transmission error is:

\[ (\dot{x}_1 - \dot{x}_2) \frac{I_1 I_2}{R_1^2 I_2 + R_2^2 I_1} + c(\dot{x}_1 - \dot{x}_2) + k(x_1 - x_2) = W \]  
\[ (42) \]

where \( W = \frac{T_1}{R_1} = \frac{T_2}{R_2} \) is the transmitted load.

Lets us introduce an equivalent-mass of the dynamical system (42):

\[ m = \frac{I_1 I_2}{R_1^2 I_2 + R_2^2 I_1} \]  
\[ (43) \]

and the transmission error \( z \) along the line of action:

\[ z = x_1 - x_2 \]  
\[ (44) \]

The resulting equation is:

\[ m\ddot{z}(t) + c(t)\dot{z}(t) + k(t)z(t) = W \]  
\[ (45) \]

In this case \( z \) represents the transmission error due to deformation between the gears during meshing. In eq. (45) both damping and stiffness coefficients are time variable. In the following the damping effect is neglected; indeed, the evaluation of \( c(t) \) is beyond the purpose of the present work.

**Stiffness evaluation**

A 2D finite elements analysis is applied to calculate stiffness. MSC-MARC 2003 is used to perform the analysis. In the analysis the whole gears are considered, neglecting teeth which are not involved in the meshing. In the present case only three teeth per gear are modeled, without loss of accuracy. More than 2400 quadrilateral elements (MARC Quad mesh) are used for each gear (fig. 10). The mesh of the teeth is more refined than the mesh of the wheels bodies, because the deformation is almost completely localized to the teeth.

**Fig. 10: Mesh of the fem model**

The driven wheel shaft is considered locked, i.e. the inner circumference of this wheel has zero rotation. This means that the driven wheel can be deformed around its shaft. The shaft elasticity is not considered here. The driver wheel can rotate around its shaft, i.e. its internal circumference has only rigid rotation.

A torque of \( C=470.71878 \) Nm is applied to the driver wheel through its shaft The wheel material is an alloy steel with a Young modulus of \( E=2.06*10^{11} \) N/m² and Poisson ratio \( \nu=0.3 \).

Using a center distance of 111 mm the contact ratio becomes \( \varepsilon_\alpha=1.6 \) [16] with period mesh angle \( \theta \) equal to 12.857°. Fifteen positions are analyzed in order to obtain the driver shaft rotation \( \delta \). According to the contact ratio \( \varepsilon_\alpha=1.6 \), for the first seven positions two pairs of teeth are in contact, while for the remaining positions one pair of teeth is in contact. In the contact regions, see fig. 11, “contact elements” have been used.
The stiffness $k$ is evaluated for each position by using the following relationship which also transforms the torsion stiffness, $C/\delta_i$, into the linear stiffness by dividing it by the square of the base circle radius of the pinion:

$$k_i = \frac{C}{R_i^2 \delta_i} \quad i=1,..,15 \quad (46)$$

The stiffness $k$ depends on the angular position $\vartheta$ of the driver wheel, its period is $2\pi/\vartheta$. This periodic function is expanded as Fourier series; the angle $\vartheta$ is normalized in order to have a period angle equal to $2\pi$:

$$\vartheta = \frac{\vartheta}{2\pi} \quad (47)$$

The Fourier expansion has the following general form:

$$k(\vartheta) = \tilde{d}_0 + \sum_{i=1}^{N} \tilde{d}_i \cos(i\vartheta + \varphi_i) \quad (48)$$

where $\tilde{d}_i$ and $\varphi_i$ are Fourier expansion coefficients.

**Perturbation solution**

Equation (49) is a second order differential equation with time depending coefficients, no exact closed form solutions are available. In this paper the method of multiple scales is used to obtain an approximate solution, which furnishes instability regions of the system.

The solution is expanded in power series of a small parameter $\varepsilon$; moreover, multiple time scales are introduced in order to consider fast and slow time variations [14].

Following the procedure described by Nayfeh and Mook [14] let us introduce a set of new independent variables $T_i$ (fast and slow scales):
\[ T_0 = t \]
\[ T_1 = \varepsilon t \]
\[ T_2 = \varepsilon^2 t \]  

The derivatives with respect to \( t \) become:

\[
\begin{align*}
\frac{d}{dt} &= D_0 + \varepsilon D_1 + \varepsilon^2 D_2, \\
\frac{d^2}{dt^2} &= D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (D_1^2 + 2D_0 D_2) 
\end{align*}
\]  

(51)

where \( D_i \) are partial derivatives respect to \( T_i \).

Let us expand the solution \( z \) as a power series in \( \varepsilon \):

\[
z(t, \varepsilon) = z_0(T_0, T_1, T_2) + \varepsilon z_1(T_0, T_1, T_2) + \varepsilon^2 z_2(T_0, T_1, T_2) + \varepsilon^3 z_3(T_0, T_1, T_2) + O(\varepsilon^3)
\]  

(52)

Substituting eq. (51) and (52) into eq. (49), neglecting \( \Phi_i \) with \( i=1,..,4 \), yields to:

\[
D_0^2 z_0 + d_0 z_0 = 0
\]  

(53)

\[
D_1^2 z_1 + d_0 z_1 = -2D_0 D_1 z_0 - Pz_0
\]  

(54)

\[
D_2^2 z_2 + d_0 z_2 = -2D_0 D_2 z_1 - Pz_1 - (D_1^2 + 2D_0 D_2) z_0
\]  

(55)

\[
D_0^2 z_3 + d_0 z_3 = -2D_0 D_1 z_2 - Pz_2 - (D_1^2 + 2D_0 D_2) z_1 - (2D_0 D_3 + 2D_1 D_2) z_0
\]  

(56)

where \( P \) is:

\[
P = \cos(\omega_0 t) + \alpha \cos(2\omega_0 t) + \beta \cos(3\omega_0 t) + \gamma \cos(4\omega_0 t)
\]  

(57)

with:

\[
\varepsilon = d_1; \quad \alpha = \frac{d_2}{d_1}; \quad \beta = \frac{d_3}{d_1}; \quad \gamma = \frac{d_4}{d_1}
\]  

(58)

Equations (53), (54), (55) and (56) are obtained equating the coefficients of \( \varepsilon^0, \varepsilon^1, \varepsilon^2 \) and \( \varepsilon^3 \) to zero.

The solution of eq. (53) is:

\[
z_0(T_0, T_1, T_2) = A(T_1, T_2) e^{j\omega T_0} + \bar{A}(T_1, T_2) e^{-j\omega T_0}
\]  

(59)

where: \( d_0 = \omega^2 \), \( \bar{A} \) is the complex conjugate of \( A \).

Inserting eq. (59) into (54) we have:

\[
D_1^2 z_1 + d_0 z_1 = -2j\omega D_1 A e^{j\omega T_0}
\]

\[
-\frac{1}{2}(A e^{i(\omega_0 + \omega)T_0} + \alpha A e^{i(2\omega_0 + \omega)T_0} + \beta A e^{i(3\omega_0 + \omega)T_0} + \gamma A e^{i(4\omega_0 + \omega)T_0})
\]

\[
+ \bar{A} e^{i(\omega_0 - \omega)T_0} + \alpha \bar{A} e^{i(2\omega_0 - \omega)T_0} + \beta \bar{A} e^{i(3\omega_0 - \omega)T_0} + \gamma \bar{A} e^{i(4\omega_0 - \omega)T_0}) + C.C.
\]  

(60)

where \( C.C. \) means complex conjugates terms. At \( \varepsilon^1 \) level every harmonic generates two forcing terms whose frequencies are \((n\omega_0 - \omega)\) and \((n\omega_0 + \omega)\). The previous consideration allows to consider each harmonic contribution separately.

Secular terms

In eq. (60) the following secular term is identified:

\[
-2j\omega D_1 A e^{j\omega T_0}
\]  

(61)

Such a term gives rise to spurious divergent solutions and must be removed. Depending on the value of \( \omega \) four cases are possible:
1. \( \omega_i - \omega = \omega \rightarrow \omega_i = 2\omega; \)
2. \( 2\omega_i - \omega = \omega \rightarrow \omega_i = \omega; \)
3. \( 3\omega_i - \omega = \omega \rightarrow \omega_i = \frac{2}{3}\omega; \)
4. \( 4\omega_i - \omega = \omega \rightarrow \omega_i = \frac{\omega}{2}; \)

**Case 1: \( \omega_f = 2\omega \)**

Eliminating secular terms gives

\[
-2j\omega A e^{i\omega T_0} - \frac{1}{2}A e^{i(\omega_i - \omega) T_0} = 0
\]

(62)

Assuming \( \omega_i = \omega + \epsilon \sigma, \) where \( \sigma \) is a detuning parameter, and \( D_f A = A' \) and considering that \( T_1 = \epsilon T_0, \) eq. (62) becomes:

\[
-2j\omega A' e^{i\omega T_0} - \frac{1}{2}A e^{i\omega T_0} = 0
\]

(63)

Since \( A \) is a complex number depending on \( T_0, \) it can be expressed as follow:

\[
A(T_1) = B(T_1) e^{\frac{\epsilon T_1}{2}}
\]

(64)

where \( B(T_1) \) is a complex function, eq. (63) becomes:

\[
-2j\omega B' + \omega \sigma B - \frac{B}{2} = 0
\]

(65)

Equation (65) can be decomposed into its real and imaginary parts:

\[
\begin{cases}
2\omega B_1' + \omega \sigma B_R - \frac{B_R}{2} = 0 \\
-2\omega B_1' + \omega \sigma B_1 + \frac{B_1}{2} = 0
\end{cases}
\]

(66)

where \( B = B_1 + jB_R. \)

Assuming that \( B(T_1) = (G_R + jG_i) e^{\frac{\epsilon T_1}{2}}, \) we can write eq. (66) in a matrix form:

\[
\xi
\begin{bmatrix}
0 & 2\omega \\
-2\omega & 0
\end{bmatrix}
\begin{bmatrix}
G_R \\
G_i
\end{bmatrix}
+ \begin{bmatrix}
\omega \sigma - \frac{1}{2} \\
0
\end{bmatrix}
\begin{bmatrix}
G_R \\
G_i
\end{bmatrix}
= 0
\]

(67)

System (66) and (67) are equivalent and the solution can be found imposing:

\[
\begin{bmatrix}
0 & 2\omega \\
-2\omega & 0
\end{bmatrix}
\begin{bmatrix}
G_R \\
G_i
\end{bmatrix}
= 0
\]

(68)

This eq. has non trivial solution if the determinant of its matrix is equal to zero, which means:

\[
\xi = \frac{1}{4\omega^2} \left( \frac{1}{4} - \omega^2 \sigma^2 \right)
\]

(69)

Instability occurs when \( \xi > 0, \) i.e. when:

\[
\left( \frac{1}{4} - \omega^2 \sigma^2 \right) > 0
\]

(70)

Remembering that \( \epsilon = d_1 \) and substituting \( \sigma = \frac{\omega_f - 2\omega}{\epsilon} \) into (70) yields:

\[
2\sqrt{d_0} - \frac{d_1}{2\sqrt{d_0}} < \omega_f < 2\sqrt{d_0} + \frac{d_1}{2\sqrt{d_0}}
\]

(71)
The same approach is applied to cases 2, 3, and 4.

Case 2: \( \omega_j = \omega \)

In this case the detuning parameter \( \sigma \) is:

\[
\sigma = \frac{2\omega_j - 2\omega}{\varepsilon}
\]

which leads to the following instability region:

\[
\sqrt{d_0} - \frac{d_3}{4\sqrt{d_0}} < \omega_j < \sqrt{d_0} + \frac{d_3}{4\sqrt{d_0}}
\]

(72)

Case 3: \( \omega_j = \frac{2}{3} \omega \)

In this case the detuning parameter \( \sigma \) is:

\[
\sigma = \frac{3\omega_j - 2\omega}{\varepsilon}
\]

which leads to the following instability region:

\[
\frac{2}{3} \sqrt{d_0} - \frac{d_3}{6\sqrt{d_0}} < \omega_j < \frac{2}{3} \sqrt{d_0} + \frac{d_3}{6\sqrt{d_0}}
\]

(73)

Case 4: \( \omega_j = \frac{\omega}{2} \)

In this case the detuning parameter \( \sigma \) is:

\[
\sigma = \frac{4\omega_j - 2\omega}{\varepsilon}
\]

which leads to the following instability region:

\[
\frac{\sqrt{d_0}}{2} - \frac{d_3}{8\sqrt{d_0}} < \omega_j < \frac{\sqrt{d_0}}{2} + \frac{d_3}{8\sqrt{d_0}}
\]

(77)

In Tab.1 the instability regions for the gear pair under investigation is presented:

<table>
<thead>
<tr>
<th>Case</th>
<th>Instability region [rad/s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>62993 &lt; \omega_j &lt; 68945</td>
</tr>
<tr>
<td>2</td>
<td>32728 &lt; \omega_j &lt; 33241</td>
</tr>
<tr>
<td>3</td>
<td>21642 &lt; \omega_j &lt; 22338</td>
</tr>
<tr>
<td>4</td>
<td>16364 &lt; \omega_j &lt; 16621</td>
</tr>
</tbody>
</table>

Tab. 1: Instability regions expressed on meshing frequency \( \omega_c \).

Considering that:

\[
\Omega_p = \frac{60\omega_j}{2\pi z_1}
\]

(78)

The instability regions are reported in Tab. 2 in terms of the driver gear velocity \( \Omega_p \).

<table>
<thead>
<tr>
<th>Case</th>
<th>Instability region [rpm]</th>
<th>( \Delta \Omega_p ) [rpm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>21484 &lt; \Omega_p &lt; 23514</td>
<td>2030</td>
</tr>
<tr>
<td>2</td>
<td>11162 &lt; \Omega_p &lt; 11337</td>
<td>175</td>
</tr>
<tr>
<td>3</td>
<td>7381 &lt; \Omega_p &lt; 7618</td>
<td>237</td>
</tr>
<tr>
<td>4</td>
<td>5581 &lt; \Omega_p &lt; 5669</td>
<td>88</td>
</tr>
</tbody>
</table>

Tab. 2: Instability regions expressed on driver gear velocity \( \Omega_p \).

CONCLUSION

In this paper instability regions for a single degrees of freedom gear model have been evaluated using multiple time scales perturbation technique. Damping, tooth separation between contact teeth and manufacturing errors
are neglected. Analytical steps allow to consider each harmonic contribution separately, by means of which four different cases are analyzed separately using different detuning parameters $\sigma$.

One case study is analyzed. Four instability regions for the driver gear velocity have been obtained, modeling teeth profiles and mesh stiffness with the approach described above. Results show instability regions occur at about 22500, 11250, 7500 and 5600 rpm. Such high values can be justified by the assumption of neglecting shafts and bearings effects on the meshing stiffness. Future development in fem model are required in order to consider the previous mentioned effects. Instability regions formulas are still applicable and can be effectively used with different values of stiffness Fourier expansion coefficients.

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REFERENCES


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