Binary Systems

Logic and Digital System Design - CS 303

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Motivation

- Analysis & Design of digital electronic circuits
- Digital circuits are used in
  - digital computers,
  - data communication,
  - digital recording,
  - digital TV,
  - and many other application require digital hardware
- Fundamental concepts in the design of digital systems
- Basic tools for the design of digital circuits
- Logic gates (AND, OR, NOT)

Digital System

- One characteristic:
- Ability of manipulating discrete elements of information
- A *set* that has a finite number of elements contains discrete information
- Examples for discrete sets
  - Decimal digits {0, 1, …, 9}
  - Alphabet {A, B, …, Y, Z}
  - Binary digits {0, 1}
- One important problem
  - how to represent the elements of discrete sets in physical systems?

How to Represent?

- In electronics circuits, we have electrical signals
  - voltage
  - current
- Different strengths of a physical signal can be used to represent elements of the discrete set.
- Which discrete set?
- Binary set is the easiest
  - two elements {0, 1}
  - Just two signal levels: 0 V and 4 V
- This is why we use binary system to represent the information in our digital system.

Binary System

- Binary set {0, 1}
  - Th elements of binary set, 0 and 1 are called *binary digits*
  - or shortly *bits*.
- How to represent the elements of other discrete sets
  - Decimal digits {0, 1, …, 9}
  - Alphabet {*A*, B, …, Y, *Z*}
- Elements of any discrete sets can be represented using <u>groups of bits.</u>
  - 9 → 1001
  - *A* → 1000001

How Many Bits?

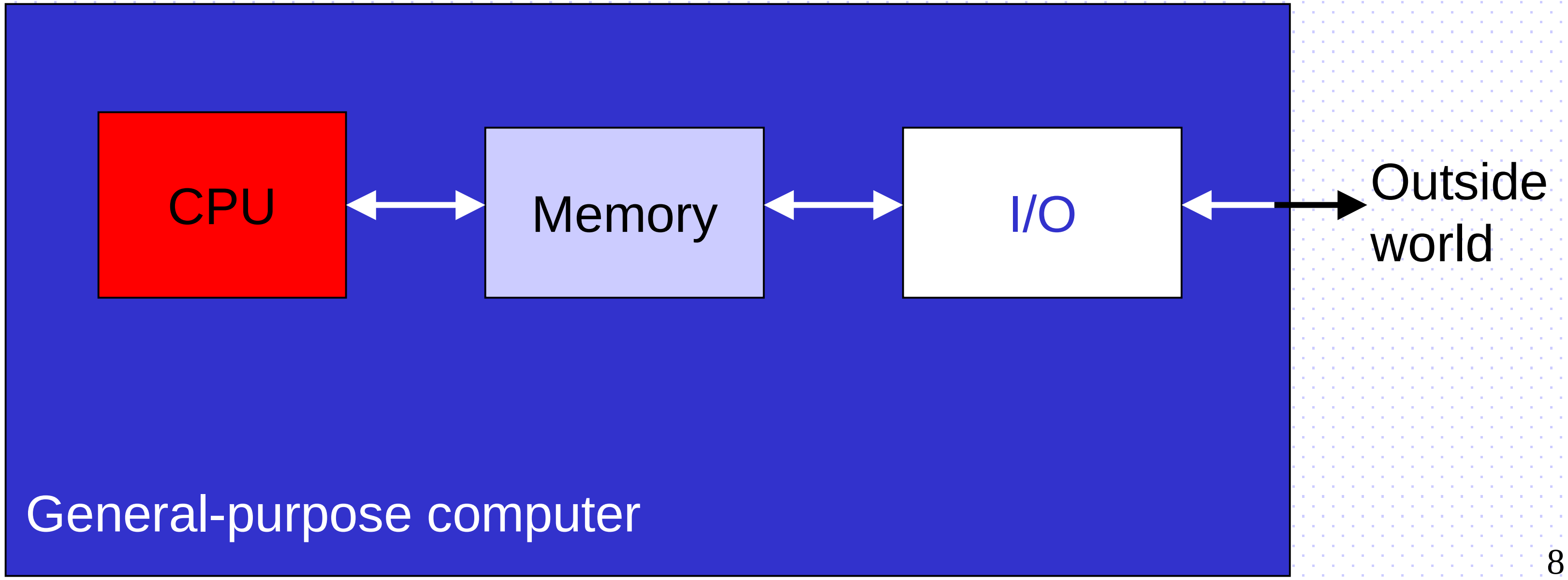
- What is the formulae for number of bits to represent a discrete set of *n* elements
- {0, 1, 2, 3}
  - 00 → 0, 01 → 1, 10 → 2, ands 11 → 3.
- {0, 1, 2, 4, 5, 6, 7}
  - 000 → 0, 001 → 1, 010 → 2, ands 011 → 3
  - 100 → 4, 101 → 5, 110 → 6, ands 111 → 7.
- The formulae, then,
  - $\lceil \log_2 n \rceil$
  - If n = 9, then $\lceil \log_2 9 \rceil = 4$.

Nature of Information

- Is information of discrete nature?
- Sometimes, but not usually.
  - Anything related to money (e.g. financial computations, accounting etc) involves discrete information
- In nature, information comes in a continuous form
  - temperature, humidity level, air pressure, etc.
- Continuous data must be converted (i.e. quantized) into discrete data
  - lose of some of the information
  - We need ADC

General-Purpose Computers

- Best known example for digital systems
- Components
  - CPU,
  - I/O units
  - Memory unit



Why Digital Systems?

- Programmable
  - underlying hardware can be used for many different applications
- Reconfigurable hardware
  - Powerful paradigm
  - (C)PLD, PLA, PAL, FPGA
- Hardware Description Languages (HDL)
  - Facilitate the use of reconfigurable hardware in more efficient way.
  - Simulation and synthesis
  - VHDL, Verilog

Anatomy of Digital Systems

- A digital system is an interconnection of digital modules
- Hierarchical structure
- Each module implements a (logical) function
- This is the essence of this class
  - to understand the logical circuits and their logical function
  - Analyze and synthesize logical circuits that are components in a digital system

Binary Numbers - 1

- Internally, information in digital systems is of binary form
  - groups of bits (i.e. binary numbers)
  - Moreover, while the information is processed, all the processing (arithmetic, logical, etc) are performed on binary numbers.
- <u>Example</u>: 4392
  - In decimal, $4392 = (4\times10^3 + 3\times10^2 + 9\times10^1 + 2\times10^0)$
  - Convention: write only the coefficients.
  - $A = a_6\, a_5\, a_4\, a_3\, a_2\, a_1\, a_0 \,.\, a_{-1}\, a_{-2}\, a_{-3}$ where $a_j \in \{0, 1, \ldots, 9\}$
  - How do you calculate the value of A?

Binary Numbers - 2

- Decimal system
  - coefficients are from {0,1, …, 9}
  - and coefficients are multiplied by powers of 10
  - base-10 or radix-10 number system
- Using the analogy, binary system {0,1}
  - base(radix)-2
- Example: 25.625
  - $25.625 = (2\times10^1 + 5\times10^0 + 6\times10^{-1} + 2\times10^{-2} + 5\times10^{-3})$
  - $25.625 = (1\times2^4 + 1\times2^3 + 0\times2^2 + 0\times2^1 + 1\times2^0 + 1\times2^{-1} + 0\times2^{-2} + 1\times2^{-3})$
  - $25.625 = (11001.101)_2$

Base-r Systems

- base-r (n, m)
  - $A = a_{n-1}\, r^{n-1} + \ldots + a_1\, r^1 + a_0\, r^0 + a_{-1}\, r^{-1} + a_{-2}\, r^{-2} + \ldots + a_{-m}\, r^{0-m}$
- Octal
  - base-8
  - digits {0,1, …, 7}
  - Example: $(31.5)_8 = 3\times8^1 + 1\times8^0 + 5\times8^{-1} = 25.625$
- Hexadecimal
  - base-16
  - digits {0,1, …, 9, A, B, C, D, E, F}
  - Example: $(19.A)_{16}$
  - $= 1\times16^1 + 9\times16^0 + A\times16^{-1} = 25.625$

Powers of 2

- $2^{10} = 1{,}024$ (K – Kilo)
- $2^{20} = 1{,}048{,}576$ (M – Mega)
- $2^{30} \rightarrow$ G – Giga
- $2^{40} \rightarrow$ T – Tera
- Examples:
  - A byte is 8-bit
  - 16 Gigabyte = $2^4 \times 2^{30} = 2^{34}$ bytes = 17,179,869,184

Arithmetic with Binary Numbers

	10101	21	augend
+	10011	19	addend
1	01000	40	sum

	10101	21	minuend
-	10011	19	subtrahend
0	00010	2	difference

			0	0	1	0	multiplicand (2)
		×	1	0	1	1	multiplier (11)
			0	0	1	0	
		0	0	1	0		
	0	0	0	0			
+ 0	0	1	0				
0	0	1	0	1	1	0	product (22)

Multiplication with Octal Numbers

				3	4	5	229	multiplicand
			×	6	2	1	401	multiplier
				3	4	5		
			7	1	2			
+	2	5	3	6				
	2	6	3	2	6	5	91,829	product

Base Conversions

- From base-r to decimal is easy
  - expand the number in power series and add all the terms
- Reverse operation is somewhat more difficult
- Simple idea: divide the decimal number successively by r and accumulate the remainders.
- If there is a fraction, then integer part and fraction part are handled separately.

Base Conversion Examples - 1

- <u>Example 1</u>: 55 (decimal to binary)
  1. $55/2 = 2 \times 27 + 1$ $a_0 = 1$
  2. $27/2 = 2 \times 13 + 1$ $a_1 = 1$
  3. $13/2 = 2 \times 6 + 1$ $a_2 = 1$
  4. $6/2 = 2 \times 3 + 0$ $a_3 = 0$
  5. $3/2 = 2 \times 1 + 1$ $a_4 = 1$
  6. $1/2 = 2 \times 0 + 1$ $a_5 = 1$
- <u>Example 2</u>: 144 (decimal to octal)
  1. $144/8 = 8 \times 18 + 0$ $a_0 = 0$
  2. $18/8 = 8 \times 2 + 2$ $a_1 = 2$
  3. $2/8 = 8 \times 0 + 2$ $a_2 = 2$
  - $144 = (220)_8$

Base Conversion Examples - 2

- Example 1: 0.6875 (decimal to binary)
  - When dealing with fractions, multiply by r until we get an integer instead of dividing by r

1. $0.6875 \times 2 = 1.3750 = 1 + 0.3750$ $a_{-1} = 1$
2. $0.3750 \times 2 = 0.7500 = 0 + 0.7500$ $a_{-2} = 0$
3. $0.7500 \times 2 = 1.5000 = 1 + 0.5000$ $a_{-3} = 1$
4. $0.5000 \times 2 = 1.0000 = 1 + 0.0000$ $a_{-4} = 1$

  - $0.6875 = (0.1011)_2$

- We are not always this lucky
- Consider the example (124.478) to octal

Base Conversion Examples - 3

- 124.478
  - Treat the integer part and fraction part separately
  - $124 = (174)_8$
  - Fraction part:

1. $0.478 \times 8 = 3.824 = 3 + 0.824$ $a_{-1} = 3$
2. $0.824 \times 8 = 6.592 = 6 + 0.592$ $a_{-2} = 6$
3. $0.592 \times 8 = 4.736 = 4 + 0.736$ $a_{-3} = 4$
4. $0.736 \times 8 = 5.888 = 5 + 0.888$ $a_{-4} = 5$
5. $0.888 \times 8 = 3.824 = 7 + 0.104$ $a_{-5} = 7$
6. $0.104 \times 8 = 0.832 = 0 + 0.832$ $a_{-6} = 0$
7. $0.832 \times 8 = 6.656 = 6 + 0.656$ $a_{-7} = 6$

- $124.478 = (174.3645706 \ldots)_8$

Conversions between Binary, Octal and Hexadecimal

- r = 2 (binary), r = 8 (octal), r = 16 (hexadecimal)

10110001101011.111100000110

010 110 001 101 011. 111 100 000 110 **26153.7406**

0010 1100 0110 1011. 1111 0000 0110 2C6B.F06

- Octal and hexadecimal representations are more compact.
- Therefore, we use them in order to communicate with computers directly using their internal representation

Complements

- Complementing is an operation on base-r numbers
- Goal: To simplify subtraction operation
  - Rather turn the subtraction operation into an addition operation
- Two types
  1. Radix complement (a.k.a. r's complement)
  2. Diminished complement (a.k.a. (r-1)'s complement)
- When r = 2
  1. 2's complement
  2. 1's complement

How to Complement?

- A number N in base-r
  1. $r^n - N$ r's complement
  2. $(r^n-1) - N$ (r-1)'s complement
  - where n is the number of bits we use
- Example: $r = 2, n = 4, N = 7$
  - $r^n = 2^4 = 16$, $r^n - 1 = 15$.
  - 2's complement of 7 → 16-7 = 9
  - 1's complement of 7 → 15-7 = 8
- Easier way to compute complements
  - $7 = (0111)_2 \rightarrow (1000)_2 + (0001)_2 = 8$ (2's complement)
  - $7 = (0111)_2 \rightarrow (1000)_2 = 8$ (1's complement)

Subtraction with Complements - 1

- Conventional subtraction
  - Borrow concept
  - When the minuend digit is smaller than the subtrahend digit, you borrow 1 from *a* digit in higher significant position
- With complements
  - M-N
  - $r^n - N$ r's complement of N
  - $M + (r^n - N) = M - N + r^n$

1. if $M \geq N$, the sum will produce a carry, that can be discarded
2. Otherwise, the sum will not produce a carry, and will be equal to $r^n - (N-M)$, which is the r's complement of N-M

Subtraction with Complements - 2

- Example:
  - X = 1010100 (84) and Y = 1000011 (67)
  - X-Y = ? and Y-X = ?

	X		1010100	
2's complement of	Y	+	0111101	
the result X – Y		1	0010001	discard carry
	Y		1000011	
2's complement of	X	+	0101100	
the result Y – X		0	1101111	no carry
			0010000	
		+	0000001	
			0010001	

Subtraction with Complements - 3

- Example: Previous example using 1's complement

X	1010100	
1's complement of Y	+ 0111100	
the result X – Y	1 0010000	discard carry
	+ 0000001	
	0010001	

Y	1000011
1's complement of X	+ 0101011
the result Y – X	0 1101110

Signed Binary Numbers

- Pencil-and-paper
  - Use symbols "+" and "-"
- We need to represent these symbols using bits
  - Convention:
    1. 0 positive
       1 negative
       - The leftmost bit position is used as a sign bit
  - In signed representation, bits to the right of sign bit is the number
  - In unsigned representation, the leftmost bit is a part of the number (i.e. the most significant bit (MSB))

Signed Binary Numbers

- Example: 5-bit numbers
  - 01011 → 11 (unsigned binary)
  - → +11 (signed binary)
  - 11011 → 27 (unsigned binary)
  - → -11 (signed binary)
  - This method is called "signed-magnitude" and is rarely used in digital systems (if at all)
- In computers, *a negative number is represented by the complement of its absolute value.*
- <u>Signed-complement</u> system
  - positive numbers have always "0" in the MSB position
  - negative numbers have always "1" in the MSB position

Signed-Complement System

- <u>Example</u>:
  - 11 = (01011)
  - How to represent –11 in 1's and 2's complements
  1. 1's complement –11 = 10100
  2. 2's complement -11 = 10100 + 00001 = 10101
  - If we use eight bit precision:
  - 11 = 00001011
  - 1's complement -11 = 11110100
  - 2's complement -11 = 11110101

Signed Number Representation

Signed magnitude		One's complement		Two's complement	
000	+0	000	+0	000	0
001	+1	001	+1	001	+1
010	+2	010	+2	010	+2
011	+3	011	+3	011	+3
100	-0	111	-0	111	-1
101	-1	110	-1	110	-2
110	-2	101	-2	101	-3
111	-3	100	-3	100	-4

- Issues: balance, number of zeros, ease of operations
- Which one is best? Why?

Which One?

- Signed magnitude:
  - Where to put the sign bit?
  - Adders may need an additional step to set the sign
  - There are two representations for 0.
- Try to subtract a large number from a smaller one.

  2 = 0 0 1 0

  5 = 0 1 0 1

  = 1 1 0 1 (the two's complement representation of −3)
- Two's complement provide a natural way to represent signed numbers (every computer today uses two's complement)
- Think that there is an infinite number of 1's in a signed number

  -3 = 1101 ≡ 11…11101
- What is 11111100?

Arithmetic Addition

- Examples:

```
+11        00001011        -11        11110101
 +9     +  00001001         +9     +  00001001
           --------                   --------
+20        00010100         -2        11111110


+11        00001011        -11        11110101
 -9     +  11110111         -9     +  11110111
           --------                   --------
 +2        00000010        -20        11101100
```

- No special treatment for sign bits

Arithmetic Overflow - 1

- In hardware, we have limited resources to accommodate numbers
  - Computers use 8-bit, 16-bit, 32-bit, and 64-bit registers for the operands in arithmetic operations.
  - Sometimes the result of an arithmetic operation get too large to fit in a register.
- Examples:

```
+2      0010      -3      1101      +2      0010
+4  +   0100      -5  +   1011      +6  +   0110
    ---------         ---------         ---------
+6      0110      -8      1000      +8  1   0000
```

Arithmetic Overflow - 2

```
-3      1101
-6 +    1010
   ___________
-9 1    0111
```

- <u>Rule</u>: If the MSB and the bits to the left of it differ, then there is an overflow

Subtraction with Negative Numbers

- Rule: is the same
- We take the 2's complement of the subtrahend
  - It does not matter if the subtrahend is a negative number.
  - $(\pm A) - (-B) = \pm A + B$

```
 -6      11111010             -6      11111010
-13  -   11110011    ==>     +13  +   00001101
                                  -------------
                              +7  1   00000111
```

- Signed-complement numbers are added and subtracted in the same way as unsigned numbers
- With the same circuit, we can do both signed and unsigned arithmetic

BCD Code - 1

- Binary Coded Decimal – BCD
  - Decimal number system is natural to human beings

Decimal Symbol	BCD Digit
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001

BCD Code - 2

- Example:
  - 429 = $(110101101)_2$ → 9 bits
  - 429 = $(0100\ 0010\ 1001)_{BCD}$ → 12 bits
- Binary numbers from 1010 to 1111 have no meaning
  - 10 = $(0001\ 0000)_{BCD}$ = $(1010)_2$
  - 14 = $(0001\ 0100)_{BCD}$ = $(1110)_2$
- BCD Addition

```
+4      0100
+5  +   0101
    ---------
+9      1001
```

```
+4      0100
+8  +   1000
    ---------
+12     1100      wrong
 +6 +   0110      correction step
    ---------
+12 1   0010
```

BCD Arithmetic

- Why we add 6 to correct in BCD arithmetic?
  - Any digit in BCD larger than (1001) must produce a carry.
  - 4-bit binary numbers produce a carry when the result is larger than (1111)

```
 +9      1001
 +9  +   1001
     ---------
+18  1   0010      a natural carry
 +6  +   0110      correction step
     ---------
+18  1   1000
```

More BCD Arithmetic

```
        1           1
  ----------------------------------
     0010        1001          0111
 +   0001        1000          0011
  ----------------------------------
     0100        0010          1010
              +  0110        + 0110
  ----------------------------------
     0100        1000          0000
```

Signed-10's Complement

- Same approach
  - MSD → 0 indicates positive numbers
  - MSD → 9 indicates negative numbers
- Example: How to represent -345 in BCD?
  - Subtract each digit from 9
  - Add 1 to the resulting number to get 10's complement
  - 0345 → 9654 + 1 = 9655 = -345
- Why bother with signed-10's complement arithmetic?
  - Some computers have special hardware to perform arithmetic in BCD directly
  - The reason being is to avoid conversion

Signed-10's Complement Arithmetic

- Example: 774-345

1	1		
0000	0111	0111	0100
+ 1001	+ 0110	0101	0101
1010	1110	1100	1001
+ 0110	0110	+ 0110	
1 0000	0100	0010	1001

Other Decimal Codes

Decimal digit	BCD 8421	2421	Excess-3	8 4 -2 -1
0	0000	0000	0011	0000
1	0001	0001	0100	0111
2	0010	0010	0101	0110
3	0011	0011	0110	0101
4	0100	0100	0111	0100
5	0101	1011	1000	1011
6	0110	1100	1001	1010
7	0111	1101	1010	1001
8	1000	1110	1011	1000
9	1001	1111	1100	1111
Unused bit combinations	1010,1011, 1100,1101, 1110,1111	0101,0110, 0111,1000, 1001,1010	0000,0001, 0010,1101, 1110,1111	0001,0010, 0011,1100, 1101,1110

Other Decimal Codes

- Weighted Codes:
  - BCD 8421, 2 4 2 1, 8 4 -2 -1
  - 2 4 2 1: Weights are (2, 4, 2, 1)
    - $9 = 1111 = 1\times2 + 1\times4 + 1\times2 + 1\times1 = 9$
    - $5 = (1011) = 1\times2 + 1\times2 + 1\times1 = 5$
    - how about (0101) ?
    - The advantage is self-complementing
    - $3 = 0011 \rightarrow 1100 = 6 \quad (9-3 = 6)$
    - $5 = 1011 \rightarrow 0100 = 4$
- Excess-3 is not weighted
  - also self-complementing

Alphanumeric Codes

- Besides numbers, we have to represent other type of information such as letters of alphabet, mathematical symbols.
- For English, alphanumeric character set includes
  - 10 decimal digits
  - 26 letters of the English alphabet (both lowercase and uppercase)
  - several special characters
- We need an alphanumeric code
  - ASCII
  - American Standard Code for Information Exchange
  - Uses 7 bits to encode 128 characters

ASCII Code

- 7 bits of ASCII Code
  - $(b_6\, b_5\, b_4\, b_3\, b_2\, b_1\, b_0)_2$
- <u>Examples</u>:
  - A → 65 = (1000001), … Z → 90 = (1011010)
  - a → 97 = (1100001), … z → 122 = (1111010)
  - 0 →48 = (0110000), … 9 → 57 = (0111001)
- 128 different characters
  - 26 + 26 + 10 = 62 (letters and decimal digits)
  - 32 special printable characters %, *, $
  - 34 special control characters (non-printable): BS, CR, etc

Representing ASCII Code

- 7-bit
- Most computers manipulate 8-bit quantity as a single unit (byte)
  - One ASCII character is stored using a byte
  - One unused bit can be used for other purposes such as representing Greek alphabet, italic type font, etc.
- The eighth bit can be used for error-detection
  - parity of seven bits of ASCII code is prefixed as a bit to the ASCII code.
  - A → (01000001) even parity
  - A → (11000001) odd parity
  - Detects one, three, and any odd number of bit errors

Binary Logic

- Deals with variables that takes on “two discrete values” and operations that assume logical meaning
- Two discrete values:
  - {true, false}
  - {yes, no}
  - {1,0}
- Binary logic is actually equivalent to what it is called “Boolean algebra”
  - Or we can say it is an implementation of Boolean algebra

Binary Variables and Operations

- We use A, B, C, x, y, z, etc. to denote binary variables
  - each can take on {0, 1}
- Logical operations
  1. AND → $x \cdot y = z$ or $xy = z$
  2. OR → $x + y = z$
  3. NOT → $\overline{x} = z$ or $x' = z$
  - For each combination of the values of x and y, there is a value of specified by the definition of the logical operation.
  - This definition may be listed in a compact form called <u>truth table</u>.

Truth Table

x	y	AND $x \cdot y$	OR $x + y$	NOT x'
0	0	0	0	1
0	1	0	1	1
1	0	0	1	0
1	1	1	1	0

Logic Gates

- Electronic circuits that operate on one or more input signals to produce an output signals
  - AND gate, OR gate, NOT gate
- These signals are electrical signals
  - voltage
  - current
- They take on either of two recognizable values
- For instance, voltage-operated circuits
  - 0V → 0
  - 4V → 1

Range of Electrical Signals

- What really matters is the range of the signal value

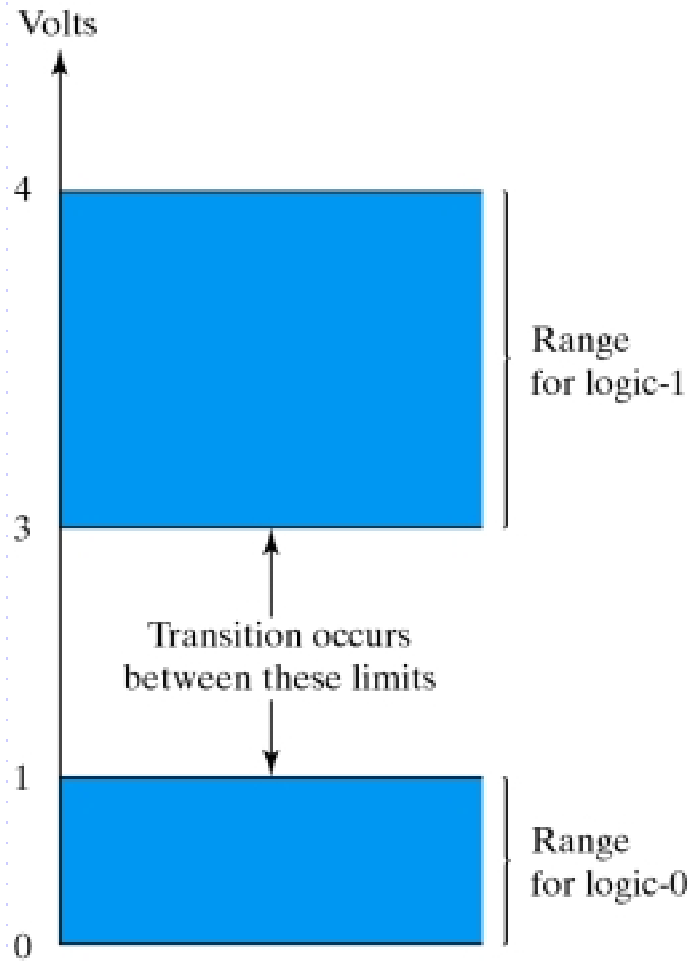
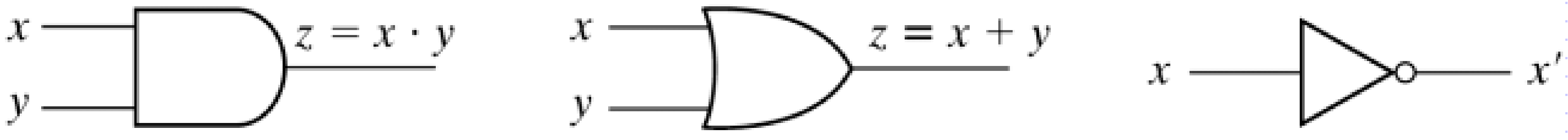


Fig. 1-3 Example of binary signals

Logic Gate Symbols



(a) Two-input AND gate (b) Two-input OR gate (c) NOT gate or inverter

Fig. 1-4 Symbols for digital logic circuits

Gates Operating on Signals

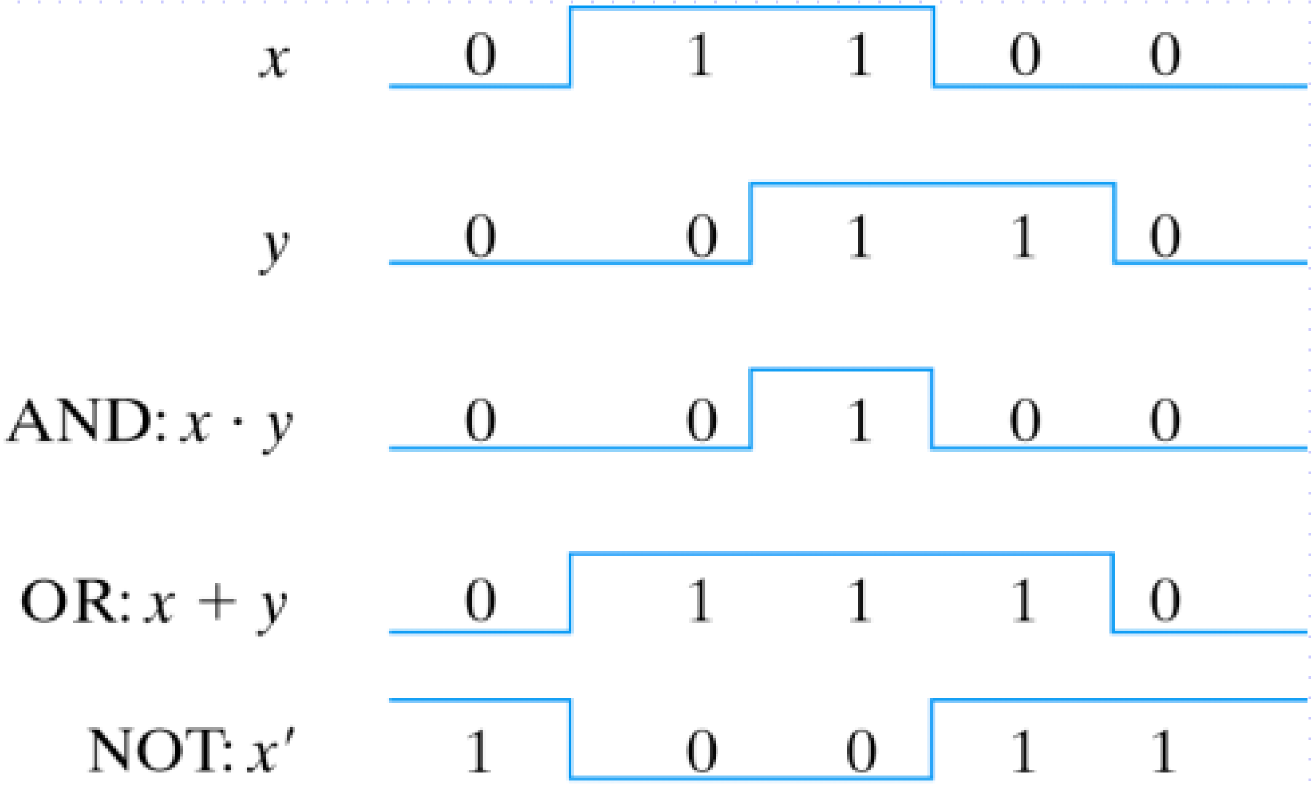
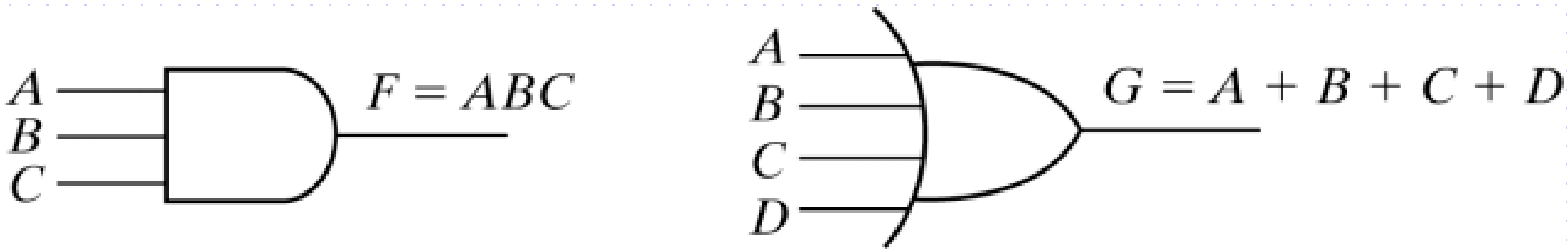


Fig. 1-5 Input-output signals for gates

Gates with More Than Two Inputs



(a) Three-input AND gate (b) Four-input OR gate

Fig. 1-6 Gates with multiple inputs