Artificial Neural Networks

Part 2/3 – Perceptron

Slides modified from Neural Network Design
by Hagan, Demuth and Beale

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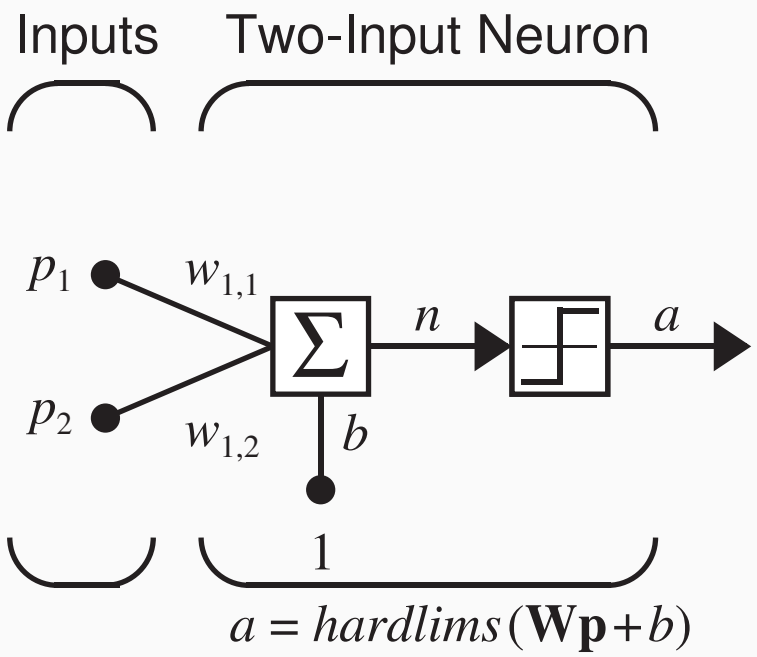
Perceptron

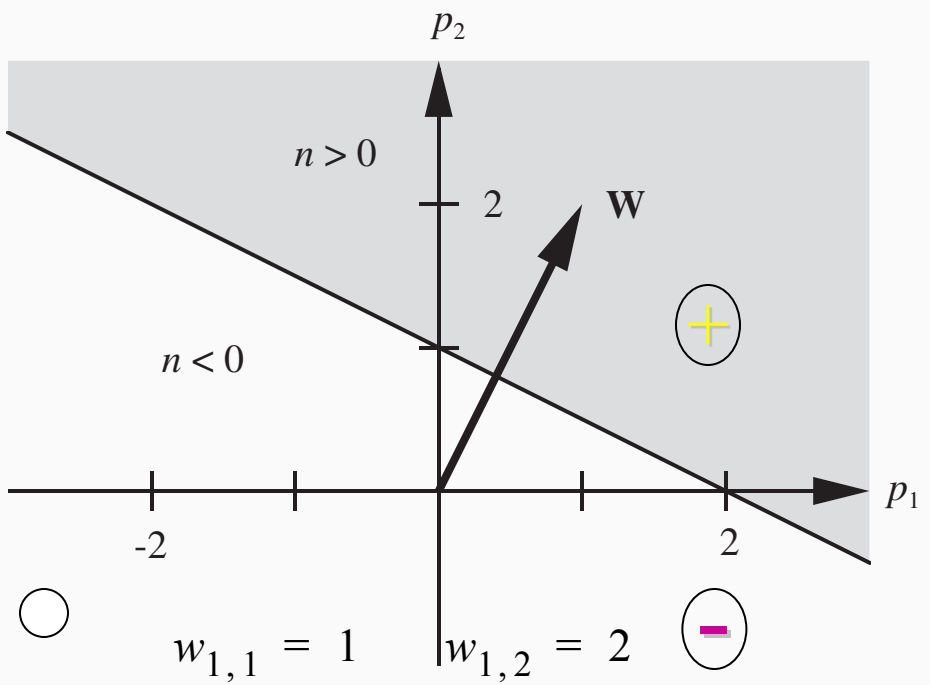
- A single artificial neuron that computes its weighted input and uses a threshold activation function.

- It effectively separates the input space into two categories by the hyperplane:

$$\mathbf{w}^T\mathbf{x} + b_i = 0$$

Two-Input Case

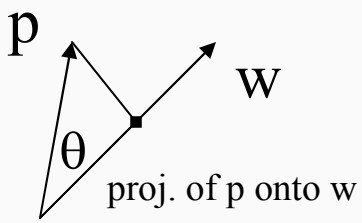




$$a = hardlims(n) = hardlims(\begin{bmatrix}1 & 2\end{bmatrix}\mathbf{p} + (-2))$$

Decision Boundary

$$\mathbf{W}\mathbf{p} + b = 0 \qquad \begin{bmatrix}1 & 2\end{bmatrix}\mathbf{p} + (-2) = 0$$



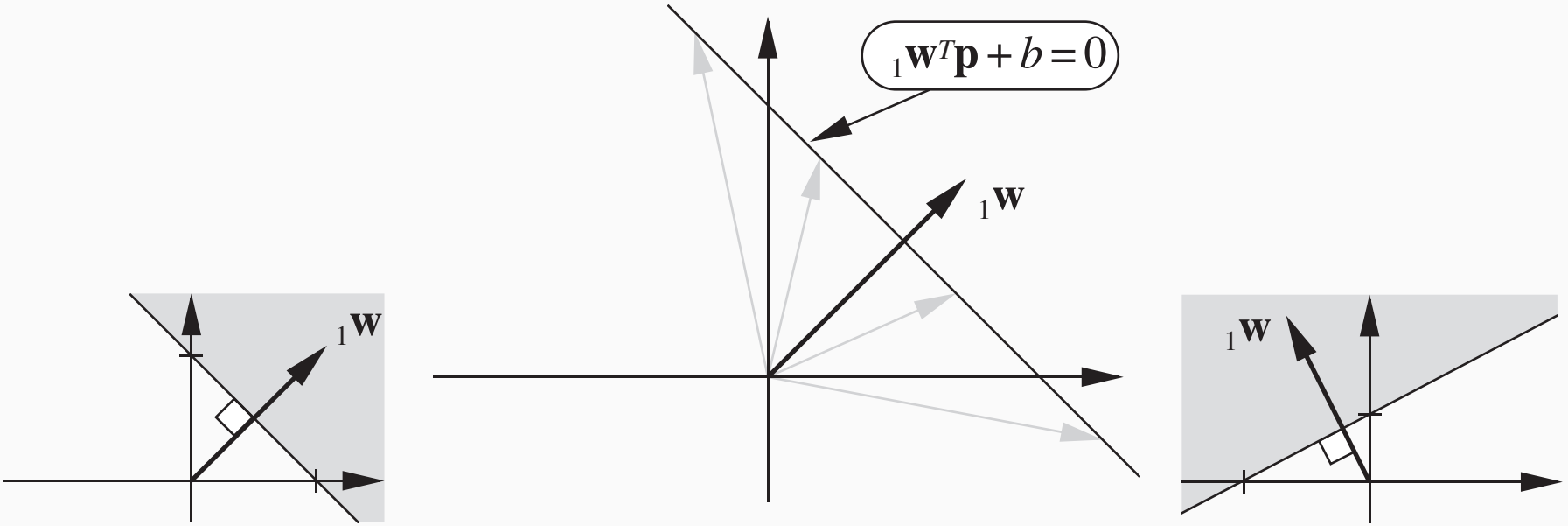
Decision Boundary

$$w^T.p = ||w||||p||\text{Cos}\theta$$

proj. of p onto w
$= ||p||\text{Cos}\theta$
$= w^T.p/||w||$

$${}_1\mathbf{w}^T\mathbf{p} + b = 0 \qquad {}_1\mathbf{w}^T\mathbf{p} = -b$$

- All points on the decision boundary have the same inner product (= -b) with the weight vector
- Therefore they have the same projection onto the weight vector; so they must lie on a line orthogonal to the weight vector



Decision Boundary

The weight vector is orthogonal to the decision boundary

The weight vector should point in the direction of the vector which should produce an output of 1

- so that the vectors with the positive output are on the right side of the decision boundary
  - if w pointed in the opposite direction, the dot products of all input vectors would have the opposite sign
  - would result in same classification but with opposite labels

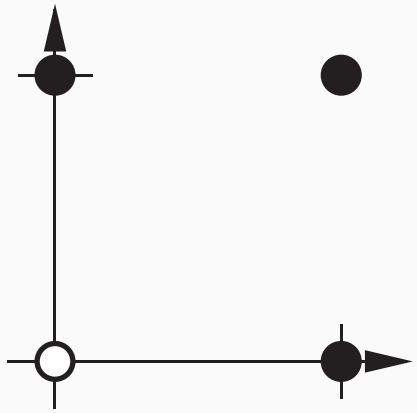
The bias determines the position of the boundary

- solve for wp+b = 0 using one point on the decision boundary to find b.

An Illustrative Example

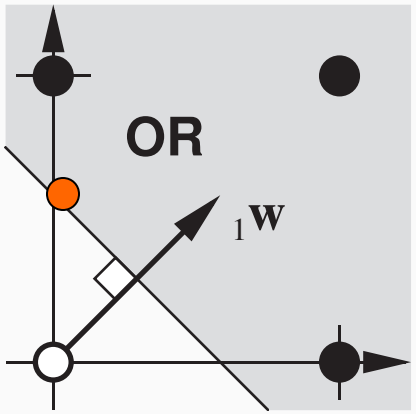
Boolean OR

$$\left\{\mathbf{p}_1 = \begin{bmatrix}0\\0\end{bmatrix}, t_1 = 0\right\} \quad \left\{\mathbf{p}_2 = \begin{bmatrix}0\\1\end{bmatrix}, t_2 = 1\right\} \quad \left\{\mathbf{p}_3 = \begin{bmatrix}1\\0\end{bmatrix}, t_3 = 1\right\} \quad \left\{\mathbf{p}_4 = \begin{bmatrix}1\\1\end{bmatrix}, t_4 = 1\right\}$$



Given the above input-output pairs (p,t), can you find (manually) the weights of a perceptron to do the job?

Boolean OR Solution



1) Pick an admissable decision boundary

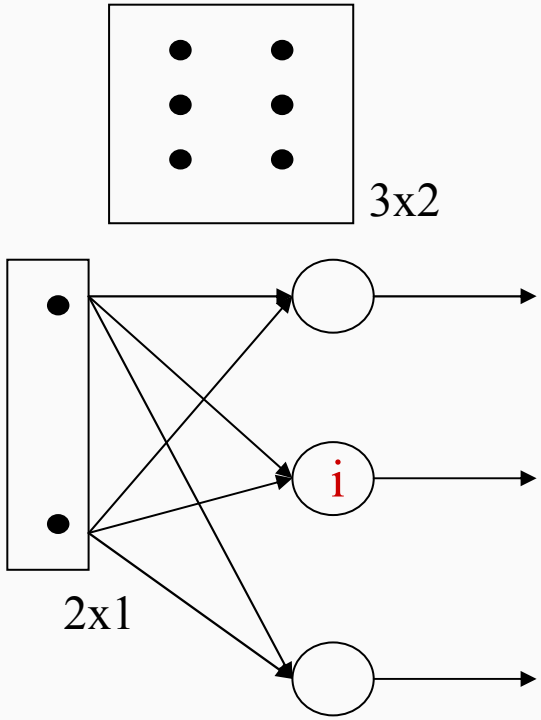
2) Weight vector should be orthogonal to the decision boundary.

$$_1\mathbf{w} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

3) Pick a point on the decision boundary to find the bias.

$$_1\mathbf{w}^T\mathbf{p} + b = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} + b = 0.25 + \mathrm{b} = 0 \quad \Rightarrow \quad b = -0.25$$

Multiple-Neuron Perceptron



$$\mathbf{W} = \begin{bmatrix} w_{1,1} & w_{1,2} & \cdots & w_{1,R} \\ w_{2,1} & w_{2,2} & \cdots & w_{2,R} \\ \vdots & \vdots & & \vdots \\ w_{S,1} & w_{S,2} & \cdots & w_{S,R} \end{bmatrix}$$

$$\mathbf{W} = \begin{bmatrix} {}_1\mathbf{w}^T \\ {}_2\mathbf{w}^T \\ \vdots \\ {}_S\mathbf{w}^T \end{bmatrix} \qquad {}_i\mathbf{w} = \begin{bmatrix} w_{i,1} \\ w_{i,2} \\ \vdots \\ w_{i,R} \end{bmatrix}$$

$$a_i = hardlim(n_i) = hardlim({}_i\mathbf{w}^T\mathbf{p} + b_i)$$

Multiple-Neuron Perceptron

Each neuron will have its own decision boundary.

$$_i\mathbf{w}^T\mathbf{p}+b_i=0$$

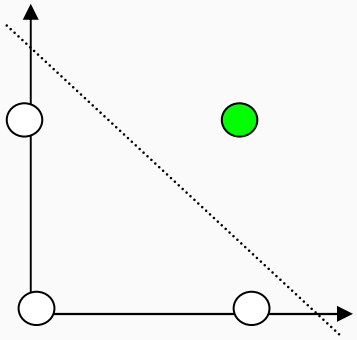
A single neuron can classify input vectors into two categories.

An S-neuron perceptron can potentially classify input vectors into 2^S categories.

Perceptron Limitations

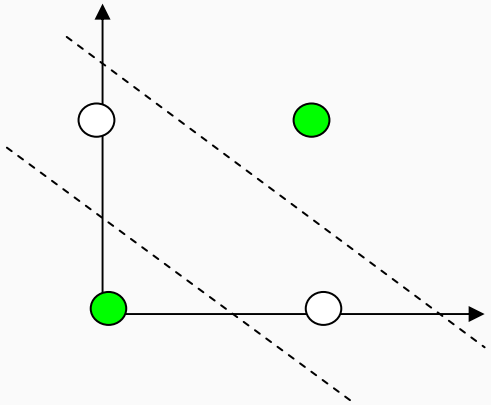
Perceptron Limitations

- A single layer perceptron can only learn linearly separable problems.
  - Boolean AND function is linearly separable, whereas Boolean XOR function is not.



Boolean AND

✓



Boolean XOR

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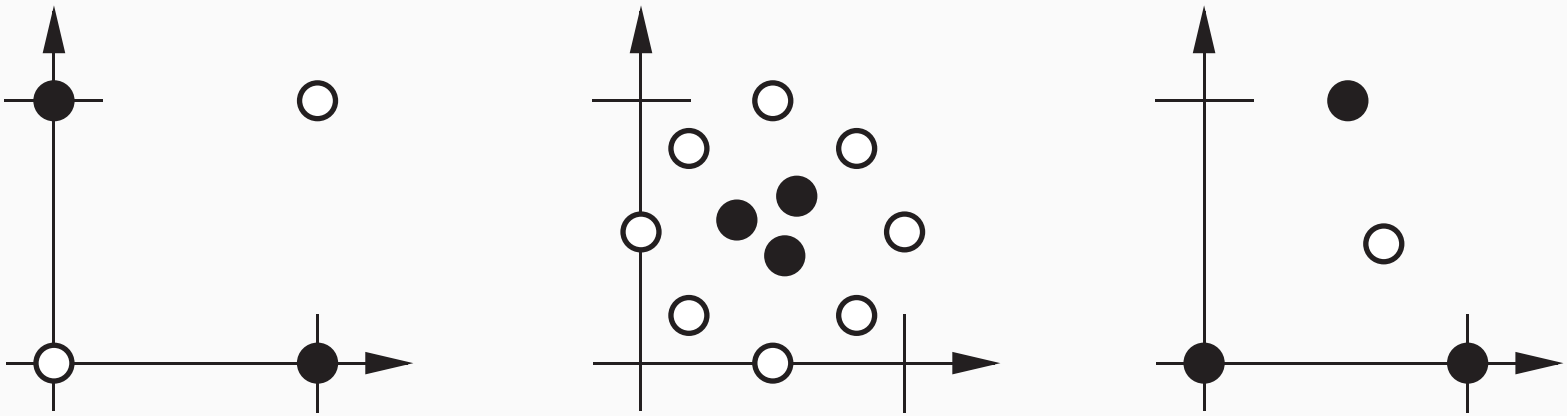
Perceptron Limitations

Linear Decision Boundary

$$_1\mathbf{w}^T\mathbf{p} + b = 0$$

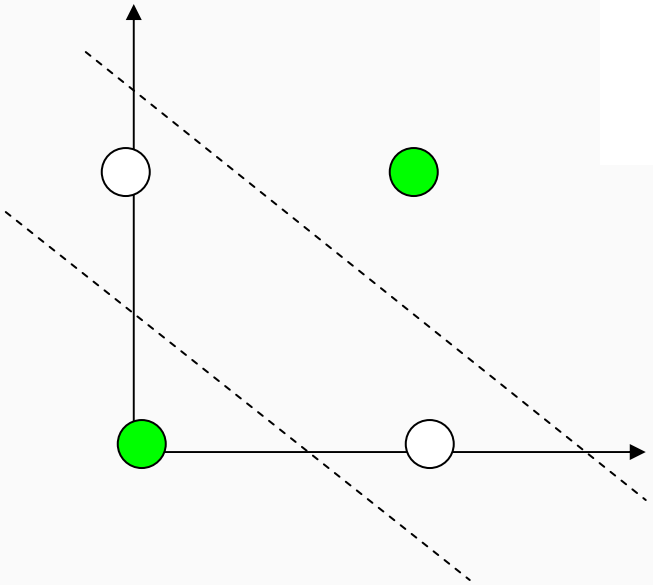
Linearly Inseparable Problems



Perceptron Limitations

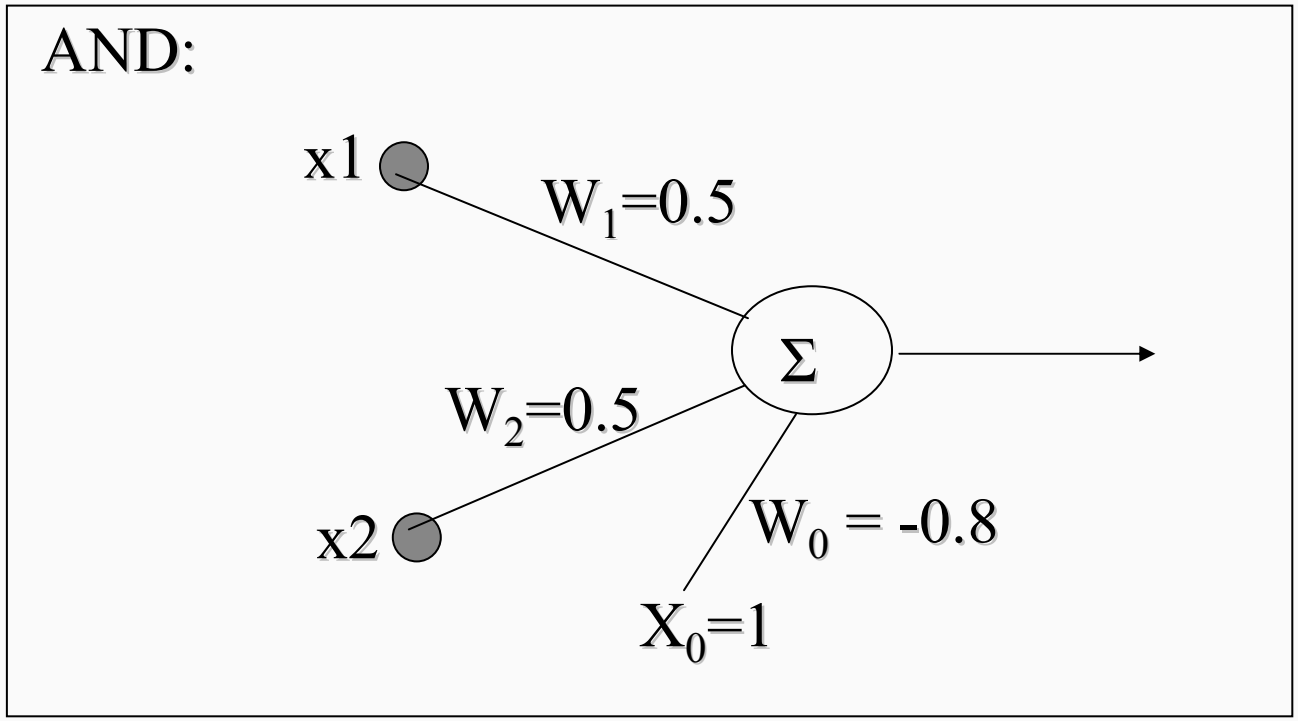
For a linearly not-separable problem:

- Would it help if we use more layers of neurons?
- What could be the learning rule for each neuron?
- Yes!



Solution: Multilayer networks and the backpropagation learning algorithm

- Perceptrons (in this context of limitations, the word refers to single layer perceptron) can learn many Boolean functions:
  - AND, OR, NAND, NOR, but not XOR
- Multi-layer perceptron can solve the XOR problem



- More than one layer of perceptrons (with a hardlimiting activation function) can learn any Boolean function.
- However, a learning algorithm for multi-layer perceptrons has not been developed until much later
  - backpropagation algorithm
  - replacing the hardlimiter in the perceptron with a sigmoid activation function

Outline

- So far we have seen how a single neuron with a threshold activation function separates the input space into two.
- We also talked about how more than one nodes may indicate convex (open or closed) regions
- The next slides explain how the weights of a perceptron can be automatically learned, using supervised learning.
  - Perceptron learning can be implemented automatically via backpropagation algorithm which we will cover in the next lecture slides

Perceptron Learning Rule

Types of Learning

- Supervised Learning

  Network is provided with a set of examples of proper network behavior (inputs/targets)

$$\{\mathbf{p}_1,\mathbf{t}_1\}, \{\mathbf{p}_2,\mathbf{t}_2\}, \ldots, \{\mathbf{p}_Q,\mathbf{t}_Q\}$$

- Reinforcement Learning

  Network is only provided with a grade, or score, which indicates network performance

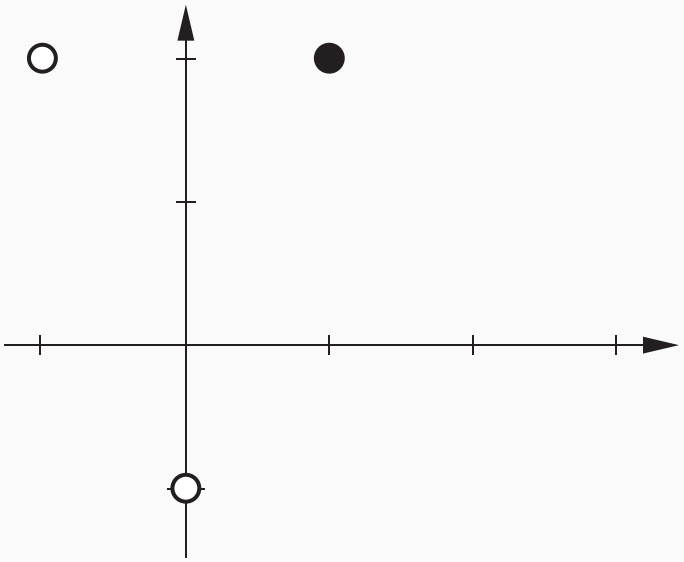
- Unsupervised Learning

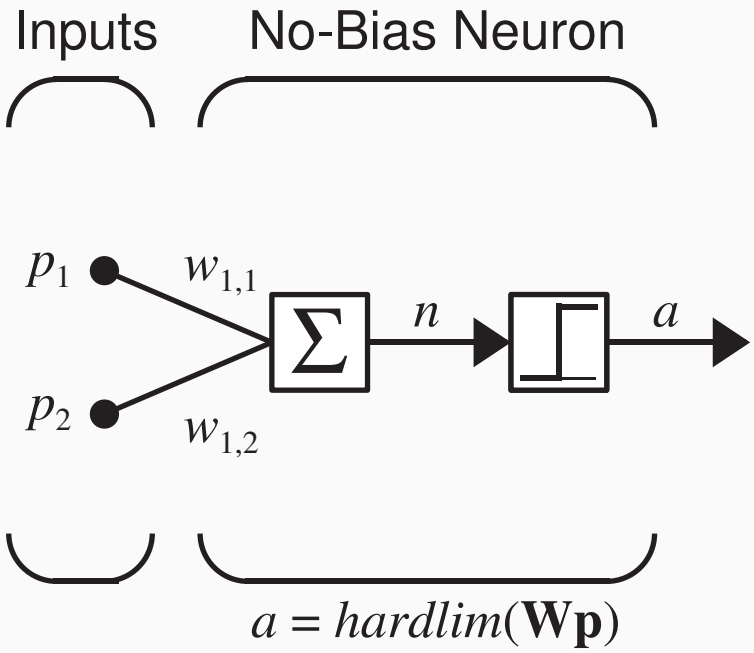
  Only network inputs are available to the learning algorithm. Network learns to categorize (cluster) the inputs.

Learning Rule Test Problem

Input-output: $\{\mathbf{p}_1, \mathbf{t}_1\}, \{\mathbf{p}_2, \mathbf{t}_2\}, \ldots, \{\mathbf{p}_Q, \mathbf{t}_Q\}$

$$\left\{\mathbf{p}_1 = \begin{bmatrix}1\\2\end{bmatrix}, t_1 = 1\right\} \quad \left\{\mathbf{p}_2 = \begin{bmatrix}-1\\2\end{bmatrix}, t_2 = 0\right\} \quad \left\{\mathbf{p}_3 = \begin{bmatrix}0\\-1\end{bmatrix}, t_3 = 0\right\}$$

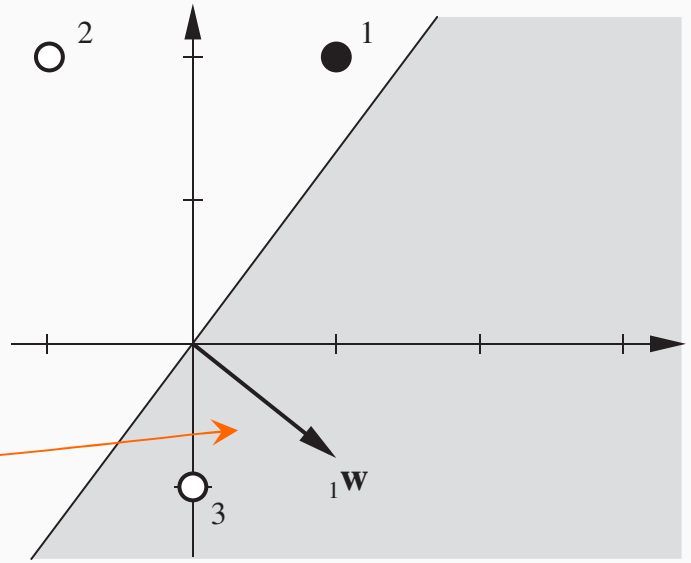




Starting Point

Random initial weight:

$$_1\mathbf{w} = \begin{bmatrix} 1.0 \\ -0.8 \end{bmatrix}$$



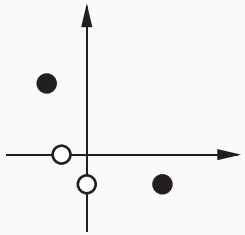
Present $\mathbf{p}_1$ to the network:

$$a = hardlim(_1\mathbf{w}^T\mathbf{p}_1) = hardlim\left(\begin{bmatrix} 1.0 & -0.8 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$$

$$a = hardlim(-0.6) = 0$$

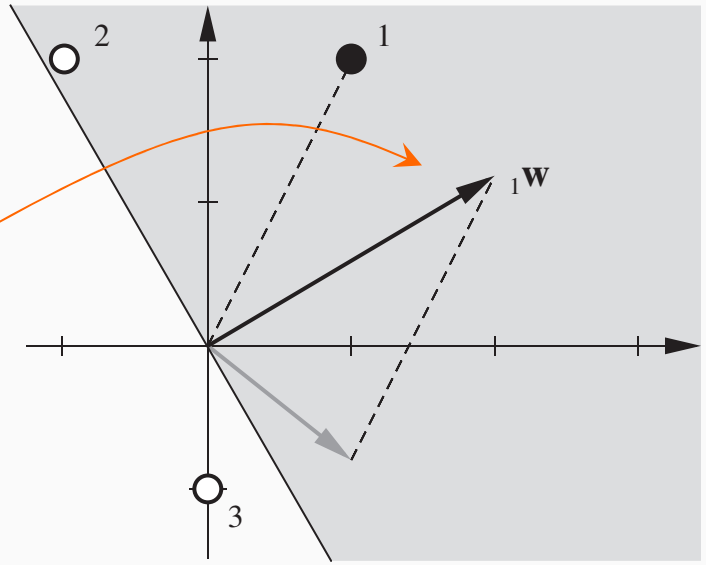
Incorrect Classification.

Tentative Learning Rule



Tentative Rule: If $t = 1$ and $a = 0$, then ${}_1\mathbf{w}^{new} = {}_1\mathbf{w}^{old} + \mathbf{p}$

$${}_1\mathbf{w}^{new} = {}_1\mathbf{w}^{old} + \mathbf{p}_1 = \begin{bmatrix} 1.0 \\ -0.8 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2.0 \\ 1.2 \end{bmatrix}$$



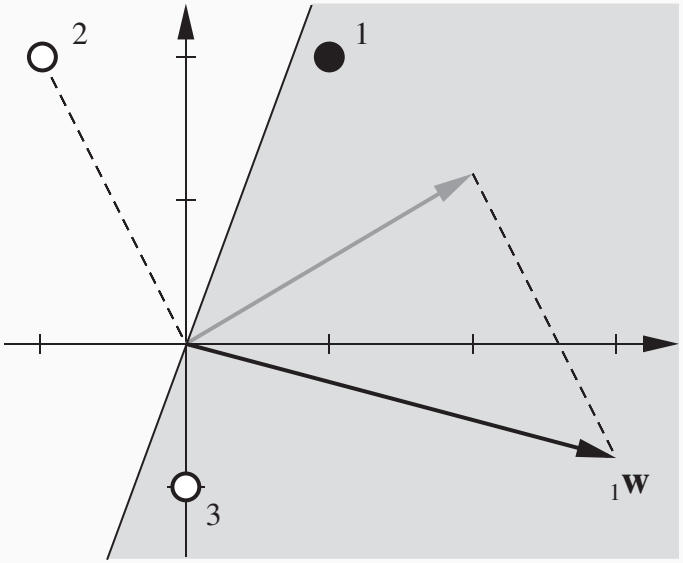
Second Input Vector

$$a = hardlim({}_1\mathbf{w}^{\mathrm{T}}\mathbf{p}_2) = hardlim\left(\begin{bmatrix}2.0 & 1.2\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}\right)$$

$$a = hardlim(0.4) = 1 \qquad \text{(Incorrect Classification)}$$

Modification to Rule: If $t = 0$ and $a = 1$, then ${}_1\mathbf{w}^{new} = {}_1\mathbf{w}^{old} - \mathbf{p}$

$${}_1\mathbf{w}^{new} = {}_1\mathbf{w}^{old} - \mathbf{p}_2 = \begin{bmatrix}2.0\\1.2\end{bmatrix} - \begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}3.0\\-0.8\end{bmatrix}$$

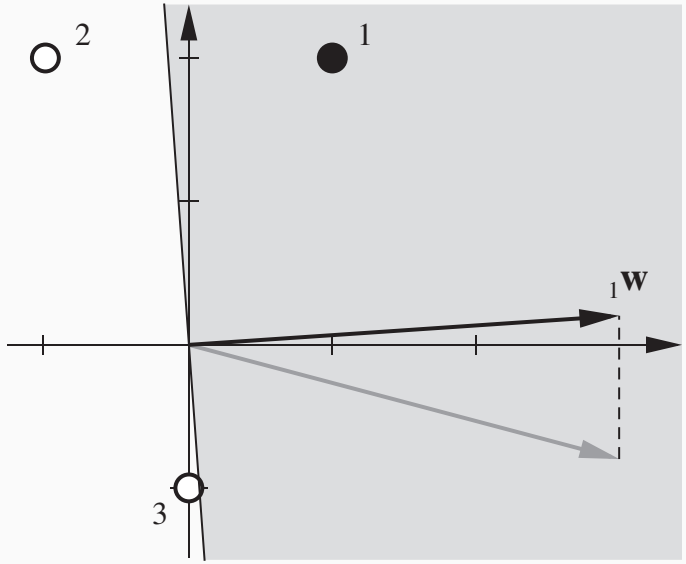


Third Input Vector

$$a = hardlim({}_1\mathbf{w}^T \mathbf{p}_3) = hardlim\left(\begin{bmatrix}3.0 & -0.8\end{bmatrix}\begin{bmatrix}0\\-1\end{bmatrix}\right)$$

$$a = hardlim(0.8) = 1 \qquad \text{(Incorrect Classification)}$$

$${}_1\mathbf{w}^{new} = {}_1\mathbf{w}^{old} - \mathbf{p}_3 = \begin{bmatrix}3.0\\-0.8\end{bmatrix} - \begin{bmatrix}0\\-1\end{bmatrix} = \begin{bmatrix}3.0\\0.2\end{bmatrix}$$



Patterns are now correctly classified.

If $t = a$, then ${}_1\mathbf{w}^{new} = {}_1\mathbf{w}^{old}$.

Unified Learning Rule

If $t = 1$ and $a = 0$, then ${}_1\mathbf{w}^{new} = {}_1\mathbf{w}^{old} + \mathbf{p}$

If $t = 0$ and $a = 1$, then ${}_1\mathbf{w}^{new} = {}_1\mathbf{w}^{old} - \mathbf{p}$

If $t = a$, then ${}_1\mathbf{w}^{new} = {}_1\mathbf{w}^{old}$

Unified Learning Rule

If $t = 1$ and $a = 0$, then ${}_1\mathbf{w}^{new} = {}_1\mathbf{w}^{old} + \mathbf{p}$

If $t = 0$ and $a = 1$, then ${}_1\mathbf{w}^{new} = {}_1\mathbf{w}^{old} - \mathbf{p}$

If $t = a$, then ${}_1\mathbf{w}^{new} = {}_1\mathbf{w}^{old}$

Define: $e = t - a$

If $e = 1$, then ${}_1\mathbf{w}^{new} = {}_1\mathbf{w}^{old} + \mathbf{p}$

If $e = -1$, then ${}_1\mathbf{w}^{new} = {}_1\mathbf{w}^{old} - \mathbf{p}$

If $e = 0$, then ${}_1\mathbf{w}^{new} = {}_1\mathbf{w}^{old}$

$\Rightarrow$

$$ {}_1\mathbf{w}^{new} = {}_1\mathbf{w}^{old} + e\mathbf{p} = {}_1\mathbf{w}^{old} + (t-a)\mathbf{p} $$

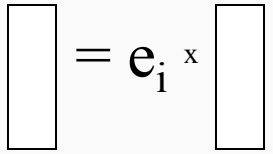
$$ b^{new} = b^{old} + e $$

A bias is a weight with an input of 1.

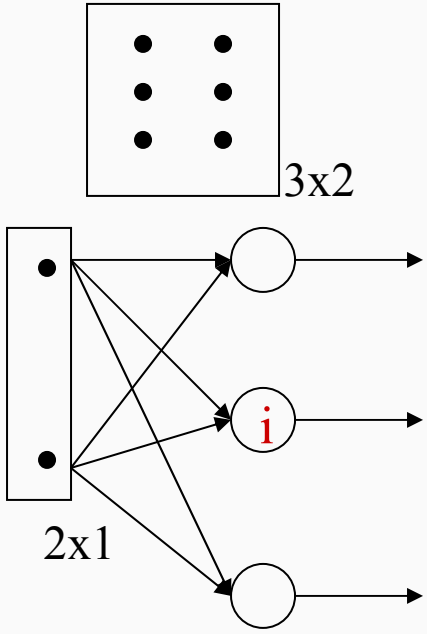
Multiple-Neuron Perceptrons

To update the ith row of the weight matrix:

$$ {}_i\mathbf{w}^{new} = {}_i\mathbf{w}^{old} + e_i \mathbf{p} $$

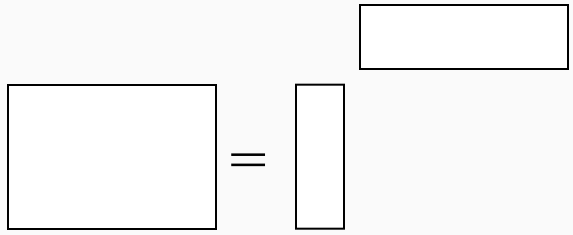


$$ b_i^{new} = b_i^{old} + e_i $$



Matrix form:

$$ \mathbf{W}^{new} = \mathbf{W}^{old} + \mathbf{e}\mathbf{p}^T $$



$$ \mathbf{b}^{new} = \mathbf{b}^{old} + \mathbf{e} $$

You should not need it, but if you were to write your own NN toolbox, you need to use matrices in order to greatly improve speed compared to a dummy algorithm working with individual neurons.

Perceptron Learning Rule (Summary)

How do we find the weights using a learning procedure?

1 - Choose initial weights randomly

2 - Present a randomly chosen pattern **x**

3 - Update weights using Delta rule:

$$w_{ij}(t+1) = w_{ij}(t) + err_i * x_j$$

where $err_i = (target_i - output_i)$

4 - Repeat steps 2 and 3 until the stopping criterion (convergence, max number of iterations) is reached

Perceptron Convergence Thm.

Theorem: The perceptron rule will always converge to weights which accomplish the desired classification, assuming that such weights exist.

Apple/Banana Example - Self Study

Training Set

$$\left\{\mathbf{p}_1 = \begin{bmatrix}-1\\1\\-1\end{bmatrix}, t_1 = \begin{bmatrix}1\end{bmatrix}\right\} \qquad \left\{\mathbf{p}_2 = \begin{bmatrix}1\\1\\-1\end{bmatrix}, t_2 = \begin{bmatrix}0\end{bmatrix}\right\}$$

Random Initial Weights

$$\mathbf{W} = \begin{bmatrix}0.5 & -1 & -0.5\end{bmatrix} \qquad b = 0.5$$

First Iteration

$$a = hardlim(\mathbf{W}\mathbf{p}_1 + b) = hardlim\left(\begin{bmatrix}0.5 & -1 & -0.5\end{bmatrix}\begin{bmatrix}-1\\1\\-1\end{bmatrix} + 0.5\right)$$

$$a = hardlim(-0.5) = 0 \qquad e = t_1 - a = 1 - 0 = 1$$

$$\mathbf{W}^{new} = \mathbf{W}^{old} + e\mathbf{p}^T = \begin{bmatrix}0.5 & -1 & -0.5\end{bmatrix} + (1)\begin{bmatrix}-1 & 1 & -1\end{bmatrix} = \begin{bmatrix}-0.5 & 0 & -1.5\end{bmatrix}$$

$$b^{new} = b^{old} + e = 0.5 + (1) = 1.5$$

Second Iteration

$$a = hardlim\,(\mathbf{W}\mathbf{p}_2 + b) = hardlim\,(\begin{bmatrix}-0.5 & 0 & -1.5\end{bmatrix}\begin{bmatrix}1\\1\\-1\end{bmatrix} + (1.5))$$

$$a = hardlim\,(2.5) = 1$$

$$e = t_2 - a = 0 - 1 = -1$$

$$\mathbf{W}^{new} = \mathbf{W}^{old} + e\mathbf{p}^T = \begin{bmatrix}-0.5 & 0 & -1.5\end{bmatrix} + (-1)\begin{bmatrix}1 & 1 & -1\end{bmatrix} = \begin{bmatrix}-1.5 & -1 & -0.5\end{bmatrix}$$

$$b^{new} = b^{old} + e = 1.5 + (-1) = 0.5$$

Check

$$a = hardlim(\mathbf{W}\mathbf{p}_1 + b) = hardlim\left(\begin{bmatrix}-1.5 & -1 & -0.5\end{bmatrix}\begin{bmatrix}-1\\ 1\\ -1\end{bmatrix} + 0.5\right)$$

$$a = hardlim(1.5) = 1 = t_1$$

$$a = hardlim(\mathbf{W}\mathbf{p}_2 + b) = hardlim\left(\begin{bmatrix}-1.5 & -1 & -0.5\end{bmatrix}\begin{bmatrix}1\\ 1\\ -1\end{bmatrix} + 0.5\right)$$

$$a = hardlim(-1.5) = 0 = t_2$$

History of Artificial Neural Networks (Details)

- **McCulloch and Pitts (1943):** first neural network model
- **Hebb (1949):** proposed a mechanism for learning, as increasing the synaptic weight between two neurons, by repeated activation of one neuron by the other across that synapse (lacked the inhibitory connection)
- **Rosenblatt (1958):** Perceptron network and the associated learning rule
- **Widrow & Hoff (1960):** a new learning algorithm for linear neural networks (ADALINE)
- **Minsky and Papert (1969):** widely influential book about the limitations of single-layer perceptrons, causing the research on NNs mostly to come to an end.
- **Some that still went on:**
  - **Anderson, Kohonen (1972):** Use of ANNs as associative memory
  - **Grossberg (1980):** Adaptive Resonance Theory
  - **Hopfield (1982):** Hopfield Network
  - **Kohonen (1982):** Self-organizing maps
- **Rumelhart and McClelland (1982):** Backpropagation algorithm for training multilayer feed-forward networks. Started a resurgence on NN research again.