Artificial Neural Networks

Part 2/3 – Perceptron

Slides modified from Neural Network Design
by Hagan, Demuth and Beale

Berrin Yanikoglu

Perceptron

- A single artificial neuron that computes its weighted input and uses a threshold activation function.

- It effectively separates the input space into two categories by the hyperplane:

$$\mathbf{w}^T\mathbf{x} + b = 0$$

Decision Boundary

The weight vector is orthogonal to the decision boundary

The weight vector points in the direction of the vector which should produce an output of 1

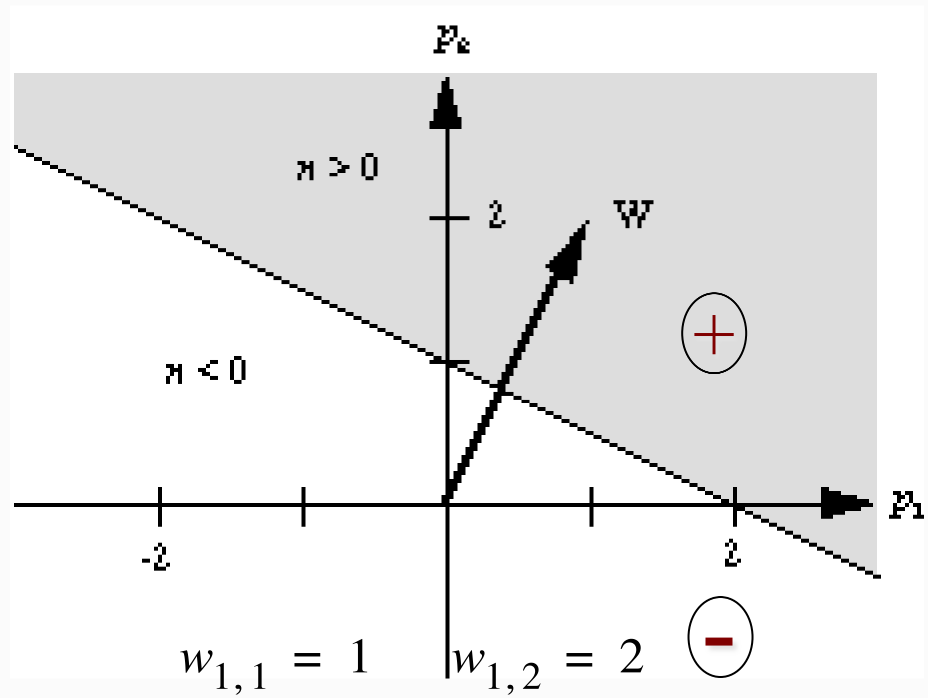
- so that the vectors with the positive output are on the right side of the decision boundary
  - if w pointed in the opposite direction, the dot products of all input vectors would have the opposite sign
  - would result in same classification but with opposite labels

The bias determines the position of the boundary

- solve for $\mathbf{w}^T\mathbf{p}+b = 0$ using *one point* on the decision boundary to find b.

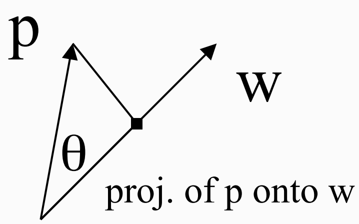
Two-Input Case

$$a = \text{hardlim}(n) = [1\ 2]\mathbf{p} + -2$$



Decision Boundary: all points p for which $\mathbf{w}^T\mathbf{p} + b = 0$

If we have the weights and not the bias, we can take a point on the decision boundary, $p=[2\ 0]^T$, and solving for $[1\ 2]\mathbf{p} + b = 0$, we see that $b=-2$.



Decision Boundary

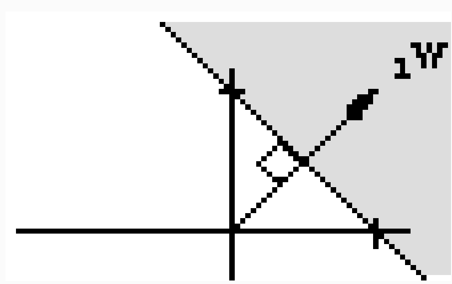
$w^T.p = ||w||||p||\cos\theta$

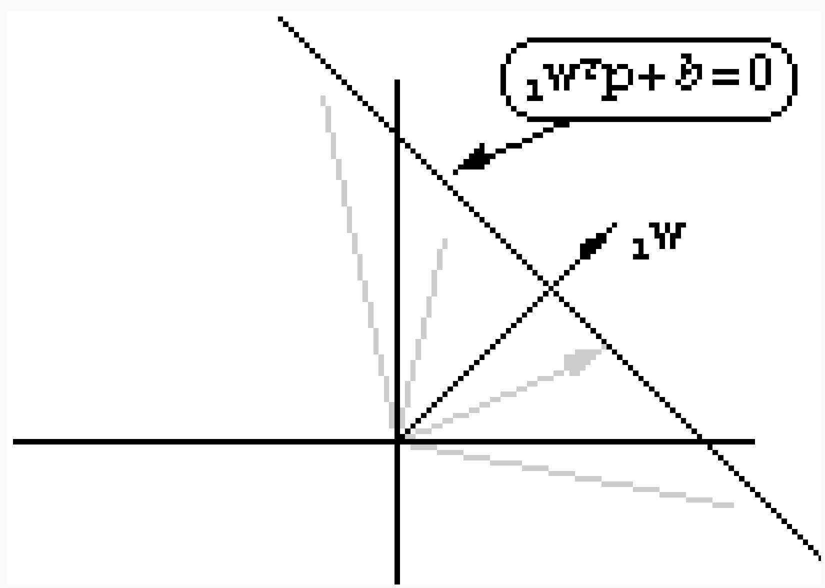
proj. of p onto w $= ||p||\cos\theta$ $= w^T.p/||w||$

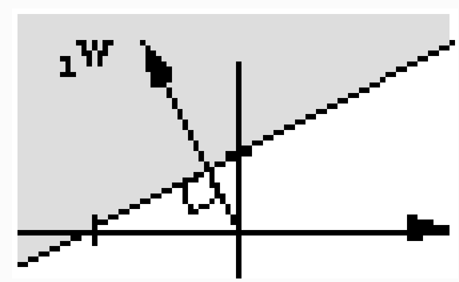
$${}_1\mathbf{w}^T\mathbf{p} + b = 0 \qquad\qquad {}_1\mathbf{w}^T\mathbf{p} = -b$$

- All points on the decision boundary have the same inner product (= -b) with the weight vector
- Therefore they have the same projection onto the weight vector; so they must lie on a line orthogonal to the weight vector

ADVANCED



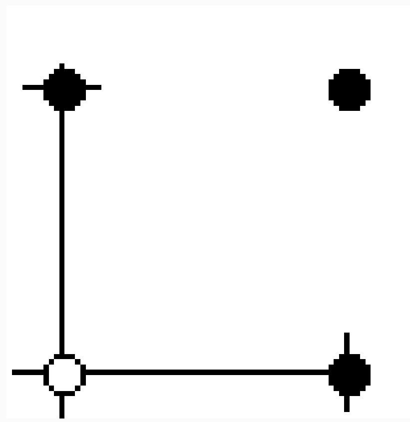




An Illustrative Example

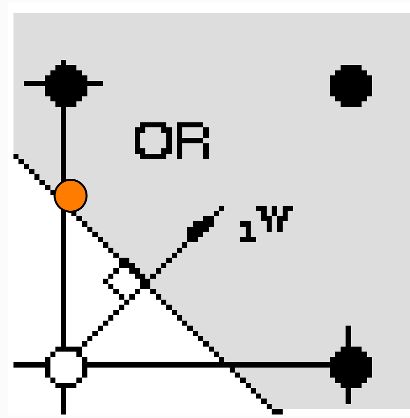
Boolean OR

$$\left\{\mathbf{p}_1 = \begin{bmatrix}0\\0\end{bmatrix}, t_1 = 0\right\} \quad \left\{\mathbf{p}_2 = \begin{bmatrix}0\\1\end{bmatrix}, t_2 = 1\right\} \quad \left\{\mathbf{p}_3 = \begin{bmatrix}1\\0\end{bmatrix}, t_3 = 1\right\} \quad \left\{\mathbf{p}_4 = \begin{bmatrix}1\\1\end{bmatrix}, t_4 = 1\right\}$$



Given the above input-output pairs (p,t), can you find (manually) the weights of a perceptron to do the job?

Boolean OR Solution



1) Pick an admissable decision boundary

2) Weight vector should be orthogonal to the decision boundary.

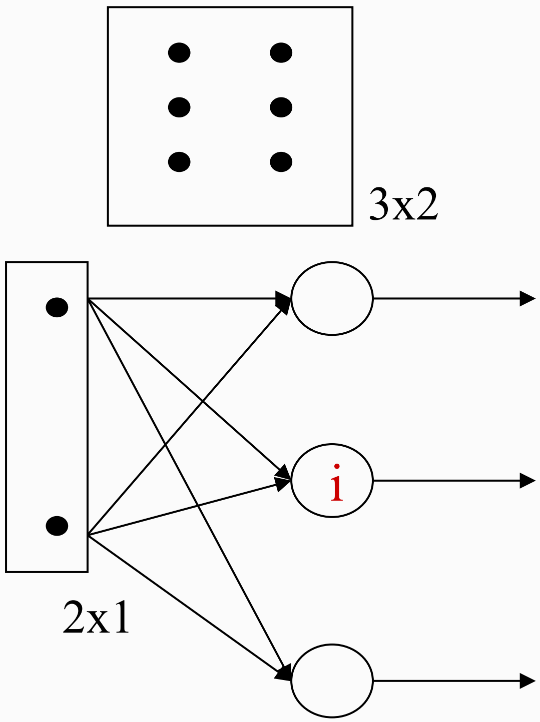
$$ {}_1\mathbf{w} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} $$

3) Pick a point on the decision boundary to find the bias.

$$ {}_1\mathbf{w}^T\mathbf{p} + b = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} + b = 0.25 + \mathrm{b} = 0 \qquad \Rightarrow \qquad b = -0.25 $$

Matrix Form

Multiple-Neuron Perceptron



weights of one neuron
in one row of W.

$$\mathbf{W} = \begin{bmatrix} w_{1,1} & w_{1,2} & \cdots & w_{1,R} \\ \boxed{w_{2,1} \quad w_{2,2} \quad \cdots \quad w_{2,R}} \\ \vdots & \vdots & & \vdots \\ w_{S,1} & w_{S,2} & \cdots & w_{S,R} \end{bmatrix}$$

$$\mathbf{W} = \begin{bmatrix} {}_1\mathbf{w}^T \\ {}_2\mathbf{w}^T \\ \vdots \\ {}_S\mathbf{w}^T \end{bmatrix} \qquad {}_i\mathbf{w} = \begin{bmatrix} w_{i,1} \\ w_{i,2} \\ \vdots \\ w_{i,R} \end{bmatrix}$$

$$a_i = hardlim(n_i) = hardlim({}_i\mathbf{w}^T\mathbf{p} + b_i)$$

Multiple-Neuron Perceptron

Each neuron will have its own decision boundary.

$$_i\mathbf{w}^T\mathbf{p} + b_i = 0$$

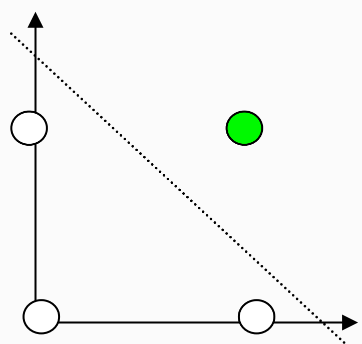
A single neuron can classify input vectors into two categories.

An S-neuron perceptron can potentially classify input vectors into 2^S categories.

Perceptron Limitations

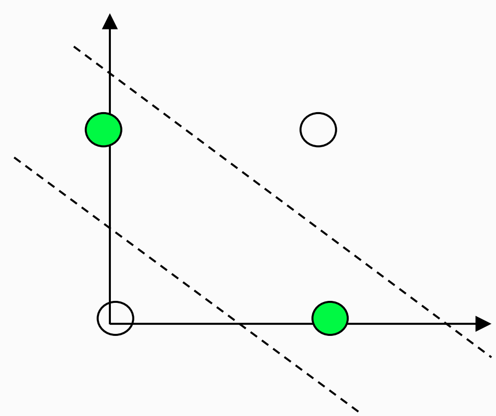
Perceptron Limitations

- A single layer perceptron can only learn linearly separable problems.
  - Boolean AND function is linearly separable, whereas Boolean XOR function is not.



Boolean AND

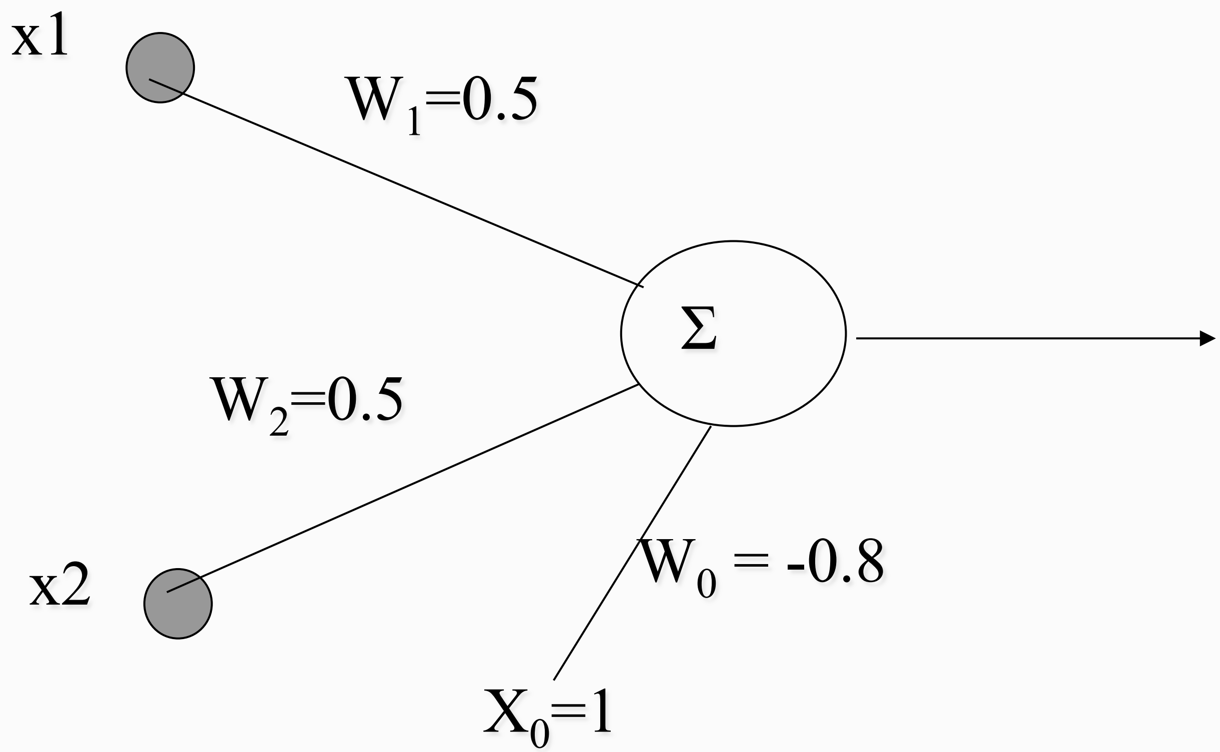
✓



Boolean XOR

✗

AND Network

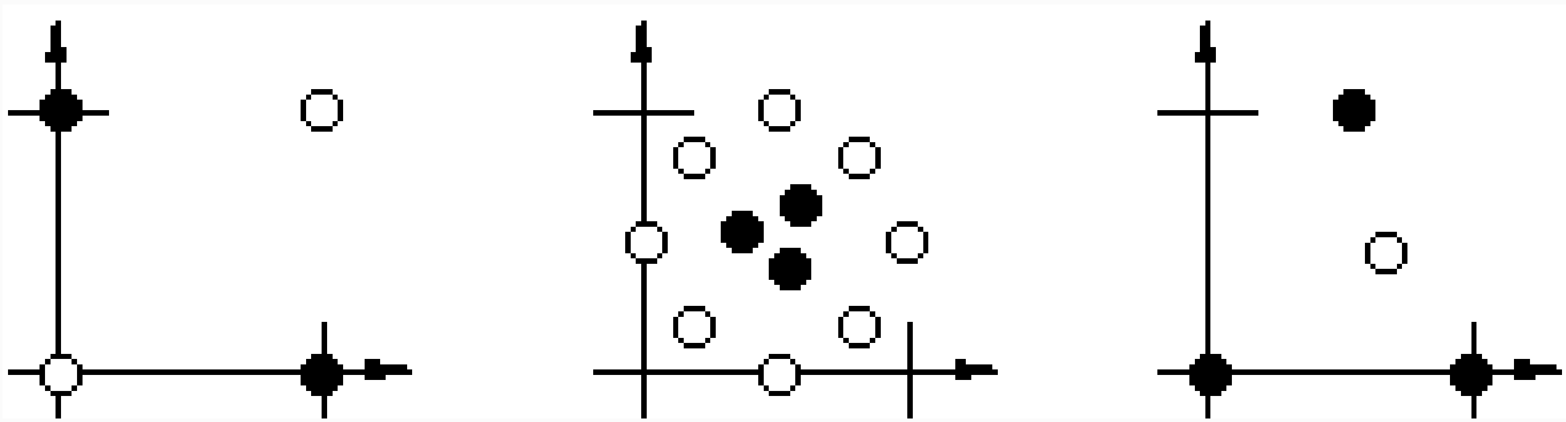


Perceptron Limitations

Linear Decision Boundary

$$ {}_1\mathbf{w}^T \mathbf{p} + b = 0 $$

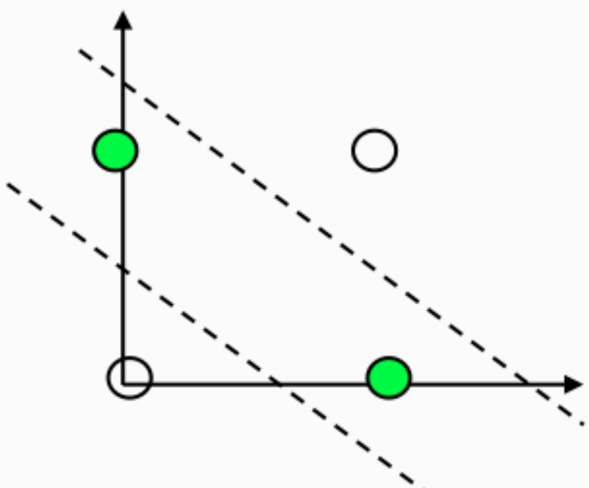
Linearly Inseparable Problems



Perceptron Limitations

For a linearly not-separable problem:

- Would it help if we use more layers of neurons?
- What could be the learning rule for each neuron?



Boolean XOR

Solution: Multilayer networks and the backpropagation learning algorithm

- More than one layer of perceptrons (with a hardlimiting activation function) can learn any Boolean function.
- However, a learning algorithm for multi-layer perceptrons has not been developed until much later
  - backpropagation algorithm
  - replacing the hardlimiter in the perceptron with a sigmoid activation function

Summary

- So far we have seen how a single neuron with a threshold activation function separates the input space into two.
- We also talked about how more than one nodes may indicate convex (open or closed) regions
- The next slides = Backpropagation algorithm to learn the weights automatically